Practice exercises 9.

1. Calculate the following limits:

a) $\lim\limits_{x\to\infty} \sin x$ b) $\lim\limits_{x\to\infty} \frac{1}{x}\sin x$ c) $\lim\limits_{x\to 0} \sin\frac{1}{x}$ d) $\lim\limits_{x\to 0} x\sin\frac{1}{x}$

2. In which points are the following functions continuous?

a) $f(x) = \begin{cases} \sin\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ b) $f(x) = \begin{cases} x\sin\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

c) $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$ d) $f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$

3. Let $f(x) = \begin{cases} \dfrac{x^2+x-2}{x^2-1} & \text{if } x \neq 1 \\ 4 & \text{if } x = 1 \end{cases}$. Is f continuous at $x_0 = 1$?

4. Choose the values of the parameters so that the following functions be continuous for all $x \in \mathbb{R}$:

a) $f(x) = \begin{cases} \dfrac{x^2+x+a}{x-3} & \text{if } x \neq 3 \\ b & \text{if } x = 3 \end{cases}$ b) $f(x) = \begin{cases} x & \text{if } |x| \leq 1 \\ x^2 + ax + b & \text{if } |x| > 1 \end{cases}$

5. Determine the points of discontinuity of the following functions. What type of discontinuities are these? (Calculate the right and left hand limits at the points to be investigated.)

a) $f(x) = \dfrac{x^2+2x-3}{x^2+5x+6}$ b) $f(x) = \dfrac{x^2-9}{x^2(x-3)^2}$ c) $f(x) = \dfrac{x^4-3x^3}{|2x^2-6x|}$

d) $f(x) = 3^{\frac{1}{x+1}}$ e) $f(x) = 3 + \dfrac{1}{1+3^{\frac{1}{1-x}}}$

6.* The Riemann function is defined as

$$f: \mathbb{R} \longrightarrow \mathbb{R},\ f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \\ \dfrac{1}{q} & \text{if } x = \dfrac{p}{q} \text{ where } p \in \mathbb{Z}, \text{ and } q \in \mathbb{N}^+ \text{ are coprimes} \end{cases}$$

Prove that

a) f is continuous at all irrational numbers;

b) f is discontinuous at all rational numbers.

7. Prove that if I is an interval and $f: I \longrightarrow \mathbb{R}$ is continuous and injective, then it is strictly monotone.

8. Is there a continuous function $f: \mathbb{R} \longrightarrow \mathbb{R}$ such that the set of zeros of f is $\left\{\frac{1}{n} : n \in \mathbb{N}^+\right\}$?

9. Prove that if $f : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous then the set of zeros of f is closed.

10. Prove that every polynomial of odd degree has a real root.

11. Let $f(x) = \frac{x^2+1}{x^2} - \frac{1}{\cos x}$. Prove that f has a zero in the open interval $\left(0, \frac{\pi}{2}\right)$.

12. Prove that there exists $x_0 \in \left[0, \frac{\pi}{2}\right]$ such that $x_0 \sin x_0 = \frac{\pi}{4}$.

13. Prove that the polynomial $f(x) = x^3 - 3x^2 - x + 2$ has 3 real roots.

14. Prove that if $f : [a, b] \longrightarrow \mathbb{R}$ is continuous and $R_f = [a, b]$, then there exists $x_0 \in [a, b]$ such that $f(x_0) = x_0$.
(Brouwer fixed-point theorem, 1-dimensional case)

15. Assuming that temperature varies continuously, prove that there are always two opposite points on the Earth with the same temperature.

16. Decide whether the functions are uniformly continuous on the given sets:

a) $f(x) = \frac{1}{x}$, $H_1 = (0, \infty)$, $H_2 = (1, \infty)$, $H_3 = [1, 2]$

b) $f(x) = x^2$, $H_1 = [0, \infty)$, $H_2 = [0, 1]$

c) $f(x) = \frac{1}{1+x^2}$, $H = [0, \infty)$

d) $f(x) = 2x^2 - x + 1$, $H = (-3, 2)$

e) $f(x) = \frac{x^3-1}{5x-5}$, $H = [-4, 1)$