Practice exercises 7.

Topology of the real numbers

1. Answer the following questions.
   (i) Find the set of interior points ($\text{int}\, H$), boundary points (∂H), limit points (H') and isolated points of H.
   (ii) Is the set H open, closed or neither? Is the set H bounded?
   (iii) Find the closure of H (it is denoted by $\overline{H}$). Is the set H compact?

a) $H = \mathbb{Z}$ b) $H = \mathbb{Q}$ c) $H = \mathbb{R} \setminus \mathbb{Q}$
d) $H = (-2, -1) \cup [3, 5] \cup \{7\} \cup [8, \infty)$
e) $H = \left\{ \frac{1}{n} : n \in \mathbb{N}+ \right\}$ f) $H = \{0\} \cup \left\{ \frac{1}{n} : n \in \mathbb{N}+ \right\}$
g) $H = \mathbb{Q} \cap [0, 1]$ h) $H = \bigcup_{n=1}^{\infty} \left[\frac{1}{2n+1}, \frac{1}{2n} \right]$

2. Prove that for any real values $a_1, \ldots, a_n$ the finite set $\{a_1, \ldots, a_n\} \subset \mathbb{R}$ is closed.

3. Let $A \subset \mathbb{R}$. Prove that
a) the set of interior points of A is open;
b) the set of boundary points of A is closed;
c) the set of limit points of A is closed.

4. Show an example of a set $A \subset \mathbb{R}$ for which $\text{int}\, \overline{A} = \mathbb{R}$ and $\overline{\text{int}\, A} = \emptyset$.

5. a) Give an example of infinitely many open sets such that their intersection is closed.
b) Give an example of infinitely many closed sets such that their union is open.

6. Show an example of an open cover of the interval $(0, 1)$ from which a finite subcover cannot be selected.

7. Prove that if G is open and F is closed then $G \setminus F$ is open and $F \setminus G$ is closed.