Practice exercises 5.

1.* Let (a_n) be a sequence of positive terms and let

$$A_n = \frac{a_1 + a_2 + \ldots + a_n}{n}, \quad G_n = \sqrt[n]{a_1 a_2 \ldots a_n}, \quad H_n = \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n}}.$$

a) Prove that if $\lim\limits_{n\to\infty} a_n = A \in \mathbb{R}$ or $\lim\limits_{n\to\infty} a_n = +\infty$ then $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} A_n = \lim\limits_{n\to\infty} G_n = \lim\limits_{n\to\infty} H_n$.

b) Using this result prove that $\lim\limits_{n\to\infty} \frac{1}{\sqrt[n]{n!}} = 0$ and $\lim\limits_{n\to\infty} \frac{n}{\sqrt[n]{n!}} = e$.

Numerical series

2. Evaluate the sum of the following series:

a) $\sum\limits_{n=1}^{\infty} \frac{1}{(3n+1)\cdot(3n+4)}$ b) $\sum\limits_{n=1}^{\infty} \frac{1}{n(n+3)}$

c) $\sum\limits_{n=1}^{\infty} \left(\sqrt{n+2} - 2\sqrt{n+1} + \sqrt{n}\right)$ d) $\sum\limits_{n=1}^{\infty} \ln\left(1 - \frac{1}{(n+1)^2}\right)$

e) $\sum\limits_{n=0}^{\infty} \frac{2^{2n}}{(-5)^{n+1}}$ f) $\sum\limits_{n=1}^{\infty} \frac{7\cdot 2^{-n} + (-3)^{n+1}}{2^{2n+1}}$ g) $\sum\limits_{n=2}^{\infty} \frac{3^{n+2} - (-2)^{n+2}}{6^n}$

3. Prove that $\sum\limits_{n=1}^{\infty} \frac{1}{n^2} < 2$.

4. Decide whether the following series are convergent or divergent (use the nth term test and the comparison test).

a) $\sum\limits_{n=1}^{\infty} \frac{n+1}{n^3-1}$ b) $\sum\limits_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n^2}\right)$ c) $\sum\limits_{n=1}^{\infty} \frac{\sin^2\left(n\sqrt{n}\right)}{n\sqrt{n}}$

d) $\sum\limits_{n=1}^{\infty} \frac{\sqrt{n+100}}{n+2}$ e) $\sum\limits_{n=1}^{\infty} \frac{1}{\sqrt[3]{2n+1}}$ f) $\sum\limits_{n=1}^{\infty} \frac{1}{\sqrt[n]{2n+1}}$

g) $\sum\limits_{n=1}^{\infty} \frac{n^2-3n+1}{n^3+2n+2}$ h) $\sum\limits_{n=1}^{\infty} \frac{2n^3+n+7}{n^5-n^2+3}$ i) $\sum\limits_{n=1}^{\infty} \frac{n^2+3n+2}{n^5-7n^3-1}$

j) $\sum\limits_{n=1}^{\infty} \frac{7n^5-2n^3+1}{n^6+2n^2-\sqrt{n}}$ k) $\sum\limits_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{\sqrt{n}}$ l) $\sum\limits_{n=1}^{\infty} \frac{2^n+3^n}{6^n+2^{n+1}}$

m) $\sum\limits_{n=1}^{\infty} \frac{2^n}{2^{n+2}-3}$ n) $\sum\limits_{n=1}^{\infty} \frac{3+7n}{5^n+n}$ o) $\sum\limits_{n=1}^{\infty} \frac{\log n}{n}$

p) $\sum\limits_{n=1}^{\infty} \frac{\log n}{n^3}$ q) $\sum\limits_{n=1}^{\infty} \frac{\log n + \sqrt{n\log n}}{n^2+1}$ r) $\sum\limits_{n=1}^{\infty} n\left(\sqrt[n]{e} - 1\right)^2$

5. Prove that there exists no real sequence $a_n > 0$ such that the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} \frac{1}{a_n}$ both converge.

6.* Using the Cauchy condensation test, investigate the convergence of the following series:

a) $\sum_{n=1}^{\infty} \frac{\log_2 n}{n}$ b) $\sum_{n=1}^{\infty} \frac{\log_2 n}{n^2}$ c) $\sum_{n=n_1}^{\infty} \frac{1}{n \cdot \log_2 n}$ d) $\sum_{n=n_1}^{\infty} \frac{1}{n \cdot (\log_2 n)^p}$

e) $\sum_{n=n_1}^{\infty} \frac{1}{n \cdot \log_2 n \cdot \log_2 \log_2 n}$ f) $\sum_{n=n_1}^{\infty} \frac{1}{n \cdot \log_2 n \cdot (\log_2 \log_2 n)^2}$

7. Estimate the error if the sum of the series is approximated by the 10th partial sum:

a) $\sum_{n=1}^{\infty} \frac{3^n}{2^{2n} + n^2 + 3}$ b) $\sum_{n=1}^{\infty} \frac{n^2 \cdot 2^{2n+2}}{(n^2 + 1) \cdot (3^{2n+1} + 5^n)}$ c) $\sum_{n=1}^{\infty} \frac{1}{n! + \sqrt{2}}$ d) $\sum_{n=1}^{\infty} \frac{n!}{(2n)!}$

8.* Using the divergence of the harmonic series, prove that
a) there are infinitely many prime numbers;
b) the series of the reciprocals of the prime numbers is divergent.