

## Practise exercises 4.

The limits  $\sqrt[n]{p} \xrightarrow{n \rightarrow \infty} 1$  ( $p > 0$ ) and  $\sqrt[n]{n} \xrightarrow{n \rightarrow \infty} 1$

1. Calculate the following limits:

$$a) a_n = \sqrt[2n]{2n}$$

$$b) a_n = \sqrt[n]{2n}$$

$$c) a_n = \sqrt[2n]{n}$$

$$d) a_n = \sqrt[5n]{3n}$$

$$e) a_n = \sqrt[n]{2n^3 + 3}$$

$$f) a_n = \sqrt[n]{\frac{2n^2 + 6}{3n^2 + 2n}}$$

$$g) a_n = \sqrt[n]{\frac{5n^2 + 4n - 5}{n^3 + 6n^2 - n}}$$

$$h) a_n = \sqrt[n^2]{n}$$

$$i) a_n = \sqrt[n]{n^3 - n^2 + 4n + 1}$$

$$j) a_n = \sqrt[n]{3^n + 2^n}$$

$$k) a_n = \sqrt[n]{3^n - 2^n}$$

$$l) a_n = \sqrt[n]{4^n + n^2 + 3n}$$

$$m) a_n = \sqrt[2n+1]{n^2 + \cos n}$$

$$n) a_n = \sqrt[n]{1 + \frac{1}{2} + \dots + \frac{1}{n}}$$

### Theorems

2. a) Prove that if for all  $n \in \mathbb{N}$   $a_n \geq 0$  and  $\lim_{n \rightarrow \infty} a_n = A$  then  $\lim_{n \rightarrow \infty} \sqrt[n]{a_n} = 1$  if  $A > 0$ .

b) Give examples to show that the limit may change if  $A = 0$ .

3. a) Prove that if  $\lim_{n \rightarrow \infty} a_n = A$  then  $\lim_{n \rightarrow \infty} a_n^n = \begin{cases} 0, & \text{if } |A| < 1 \\ \infty, & \text{if } A > 1 \\ \text{does not exist,} & \text{if } A \leq -1 \end{cases}$ .

b) Give examples to show that the limit may change or doesn't exist if  $A = 1$  or  $A = -1$ .

The limit  $(1 + \frac{x}{n})^n \xrightarrow{n \rightarrow \infty} e^x$

4. Calculate the limits of the following sequences.

$$a) a_n = \left(1 + \frac{1}{6n^2}\right)^{6n^2+2}, \quad b_n = \left(\frac{n+5}{n-4}\right)^{n+3}, \quad c_n = \left(\frac{3n+5}{3n-4}\right)^{3n}, \quad d_n = \left(\frac{3n-1}{3n+2}\right)^{2n}$$

$$b) a_n = \left(1 + \frac{1}{n}\right)^{n^2}, \quad b_n = \left(1 + \frac{1}{n^2}\right)^n, \quad c_n = \left(1 - \frac{1}{n^4}\right)^{n^3}, \quad d_n = \left(1 - \frac{1}{n^4}\right)^{n^5}$$

$$c) a_n = \left(0.9 + \frac{1}{n}\right)^n, \quad b_n = \left(2 - \frac{1}{n}\right)^n, \quad c_n = \left(\frac{4n+1}{7n+5}\right)^n, \quad d_n = \left(\frac{6n+1}{4n+5}\right)^n$$

$$d) a_n = \left(\frac{3n^2+1}{3n^2-2}\right)^{3n^2}, \quad b_n = \left(\frac{3n^2+1}{3n^2-2}\right)^{3n^3}, \quad c_n = \left(\frac{3n^2-2}{3n^2+1}\right)^{3n^3}, \quad d_n = \left(\frac{3n^2-1}{3n^2+2}\right)^{3n}$$

5. Using the formula  $\left(1 + \frac{1}{z_n}\right)^{z_n} \rightarrow e$  (for  $z_n \rightarrow \infty$ ) evaluate the following limits:

a)  $a_n = \left(\frac{2^n + 3}{2^n + 1}\right)^n$    b)  $a_n = \left(\frac{n^2 - n + 1}{n^2 + n + 1}\right)^n$    c)  $a_n = \left(\frac{n^2 + 3n - 4}{n^2 - n + 2}\right)^{4n+1}$

d)  $a_n = \left(\frac{n^2 + \sqrt{n} + 1}{n^2 - 1}\right)^{4n^2+3}$

## Recursive sequences

6. Investigate the convergence of the following sequences and calculate the limit if it exists.

a)  $a_1 = 6, a_{n+1} = 5 - \frac{6}{a_n}, n = 1, 2, \dots$

b)  $a_1 = \frac{4}{3}, a_{n+1} = \frac{3 + a_n^2}{4}, n = 1, 2, \dots$

c)  $a_1 = 1, a_{n+1} = \sqrt{6 + a_n}, n = 1, 2, \dots$

d)  $a_1 = -3, a_{n+1} = \frac{5 - 6a_n^2}{13}, n = 1, 2, \dots$

e)  $a_1 = 1, a_{n+1} = \frac{a_n}{1 + a_n}, n = 1, 2, \dots$

f)  $b_1 = 2, b_{n+1} = 4 + \sqrt{b_n - 2} - \frac{4}{\sqrt{n+4}}$

7.\* Let  $A > 0, x_1 = 1$  and  $x_{n+1} = \frac{x_n + \frac{A}{x_n}}{2}$ . Prove that  $x_n \rightarrow \sqrt{A}$ .

## Limit superior and limit inferior

8. Find the limit inferior and limit superior of the following sequences.

a)  $a_n = \frac{(-1)^n}{n} + \frac{1 + (-1)^n}{2}$

b)  $a_n = (3 + (-1)^n)n$

c)  $a_n = 1 + 2(-1)^n + 3(-1)^{\frac{n(n+1)}{2}}$

d)  $a_n = \cos\left(\frac{n\pi}{2}\right) \cdot \frac{2n^2 - 3}{n^2 + n + 8}, b_n = \cos\left(\frac{n\pi}{2}\right) \cdot \frac{2n^2 - 3}{n^3 + n + 8}$

e)  $a_n = \frac{(-3)^n + 8}{5 + 4^n}, b_n = \frac{(-4)^n + 8}{5 + 4^n}$

f)  $a_n = \sqrt{\frac{n^3 + (-1)^n n^3}{3n^3 + n + 7}}$

## Additional exercises

9)\* For all  $r > 0$  show examples for sequences  $a_n \rightarrow 0+$  and  $b_n \rightarrow 0$  such that

$$a_n^{b_n} \rightarrow r.$$

10)\* For all  $n \in \mathbb{N}$  we define the value of  $a_n$  by placing a decimal point in front of the index  $n$  written in the decimal number system and then a zero digit in front of it, and we interpret the number thus obtained in the decimal number system. For example  $a_{4523} = 0.4523$  and  $a_{100} = 0.100$ . Find the accumulation points of the number sequence  $(a_n)$ .

11)\* Consider the following number sequence:

$$\frac{0}{1}, \frac{0}{2}, \frac{1}{2}, \frac{0}{3}, \frac{1}{3}, \frac{2}{3}, \frac{0}{4}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \dots \frac{0}{n}, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}, \frac{0}{n+1}, \frac{1}{n+1}, \dots$$

- a) Is 0 an accumulation point of the sequence?
  - b) Is 1 an accumulation point of the sequence?
  - c) Is the sequence convergent?
  - d) Exactly what real numbers are the accumulation points of the sequence?
- Give reasons for your answers.

12)\* Is there a number sequence whose real accumulation points are

- a) the integers;
- b) the rational numbers;
- c) the points of the set  $\left\{1, \frac{1}{2}, \frac{1}{3}, \dots\right\}$ ;
- d) the points of the set  $\left\{1, \frac{1}{2}, \frac{1}{3}, \dots\right\} \cup \{0\}$ ;
- e) the  $[0, 1]$  closed interval?

If the answer is yes, then construct such a sequence.