

Practise exercises 3.

1. Find the limit of (a_n) and using the definition, provide a threshold index N for a given $\varepsilon > 0$.

$$a) a_n = \frac{7n+4}{2n-1}$$

$$b) a_n = \frac{3n^2+4n+7}{n^2+n+1}$$

$$c) a_n = \frac{n^2-10^8}{5n^6+2n^3-1}$$

$$d) a_n = 1 + (-1)^{n+1} \cdot 2^{3-n}$$

2. Using the definition of the limit, show that

$$a) \lim_{n \rightarrow \infty} (6n^3 + 3n) = \infty$$

$$b) \lim_{n \rightarrow \infty} \sqrt{n^2 - n} = \infty$$

$$c) \lim_{n \rightarrow \infty} (n^3 - 3n^2 + 5n + 9) = \infty$$

$$d) \lim_{n \rightarrow \infty} \frac{n^3 + 3n}{n^2 + 2} = \infty$$

$$e) \lim_{n \rightarrow \infty} \frac{1 + n^2 - 3n^3}{n^2 + 3n + 7} = -\infty$$

3. Formulate the following statements without negation:

$$a) \lim_{n \rightarrow \infty} a_n \neq A \in \mathbb{R}$$

b) (a_n) is divergent

Show that the sequence $a_n = (-1)^n$ is divergent.

4. Calculate the limit of the following sequences:

$$a) a_n = \frac{n+3}{4n^2+7n+6}$$

$$b) a_n = \frac{n-5n^4}{n^4+8n^3+1}$$

$$c) a_n = \frac{1-n^3}{70-n^2+n}$$

$$d) a_n = \frac{-n^7+n^6-3}{n^5-n^2+2}$$

$$e) a_n = \frac{(2n^3+3)^2}{(3n+6)^6}$$

$$f) a_n = \frac{(n+1)!}{(3-2n)n!}$$

$$g) a_n = \frac{\binom{n}{2}}{\binom{n}{3}}$$

$$h) a_n = \frac{\binom{2n}{4}}{\binom{n+1}{2}\binom{n-1}{2}}$$

$$i) a_n = \sqrt[3]{\frac{2n^2+6}{3n^2+2n}}$$

$$j) a_n = \frac{n^{3/2} + n^2 + 1}{\sqrt{1+n^2} + 2\sqrt{n^3+2}}$$

$$k) a_n = \frac{\sqrt[4]{n^3+6}}{\sqrt[3]{n^2+3n+2}}$$

$$l) a_n = \frac{9\sqrt[3]{n} - 3\sqrt{2n} + 1}{\sqrt[4]{n} + \sqrt{3n}}$$

5. Decide whether the following sequences converge and if so, find their limit:

$$a) a_n = \sqrt{n^2+n+1} - \sqrt{n^2-n+1}$$

$$b) a_n = \sqrt{n^2-7n+1} - \sqrt{n^2-n+4}$$

$$c) a_n = \sqrt{2n^2+3n+1} - \sqrt{n^2+1}$$

$$d) a_n = (3n+1)\left(n - \sqrt{n^2+1}\right)$$

$$e) a_n = \frac{1}{n - \sqrt{n^2+3n+5}}$$

$$f) a_n = \sqrt[3]{n^3+3n^2-1} - \sqrt[3]{n^3-2n^2+3n+2}$$

6. Prove that $\lim_{n \rightarrow \infty} \frac{a^n}{n!} = 0$, where $a \in \mathbb{R}$.

7. Calculate the limit of the following sequences:

$$a) a_n = \frac{\sin(n)}{n}$$

$$b) a_n = \frac{n^2 - 5}{2n^3 + 6n} \cos(n^4 + 5n + 8)$$

$$c) a_n = \frac{\log(n+1)}{n}$$

$$d) a_n = \frac{\log_{10}(n^2) + 3}{\log_3(n)}$$

$$e) a_n = \frac{(-3)^{n+1} + 2^{2n+3}}{8 + 5^n}$$

$$f) a_n = \frac{3^{2n} + n^2 + 1}{3^n + 9^n}$$

$$g) a_n = \frac{7^n + n^7 + 7}{4^n + 3n^2 + 5}$$

$$h) a_n = \frac{4^{n-1} + n^5 \cdot 3^{n+2}}{2^{2n+3} + 2^{n-2}}$$

$$i) a_n = \frac{n^3 2^n + 3^n}{2^{2n} - 3n^2}$$

$$j) a_n = \frac{2n! + n^{20}}{n^n}$$

$$k) a_n = \frac{(2^n + 7^n)^2}{n!}$$

8. True or false?

$$a) \text{If } a_n \rightarrow A \text{ then } a_n^2 \rightarrow A^2.$$

$$b) \text{If } a_n^2 \rightarrow A^2 \text{ then } a_n \rightarrow A.$$

$$c) \text{If } a_n > 0 \text{ and } b_n \rightarrow \infty \text{ then } a_n b_n \rightarrow \infty.$$

$$d) \text{If } a_n \rightarrow 0 \text{ then } \frac{1}{a_n} \rightarrow \infty.$$

$$e) \text{If } a_n \rightarrow \infty \text{ then } \frac{1}{a_n} \rightarrow 0.$$

$$f) \text{If } a_n > 0 \text{ and } (a_n) \text{ is convergent then } \lim_{n \rightarrow \infty} a_n > 0.$$

9. Is there a convergent sequence that can be written in the following form?

$$a) a_n + b_n \text{ where both } (a_n) \text{ and } (b_n) \text{ are divergent}$$

$$b) a_n + b_n \text{ where } (a_n) \text{ is convergent and } (b_n) \text{ is divergent}$$

$$c) a_n \cdot b_n \text{ where both } (a_n) \text{ and } (b_n) \text{ are divergent}$$

$$d) a_n \cdot b_n \text{ where } (a_n) \text{ is convergent and } (b_n) \text{ is divergent}$$

10. Prove the following statements:

$$a) \text{If } a_n \rightarrow A \text{ then } \sqrt[3]{a_n} \rightarrow \sqrt[3]{A}.$$

$$b) \text{If } c_n \rightarrow C > 0 \text{ and } d_n \rightarrow \infty \text{ then } c_n d_n \rightarrow \infty.$$