Practise exercises 2.

Supremum, infimum

1. Are the following sets bounded below or above? If so, determine their supremum and infimum. Do these sets have a minimum or a maximum?

$a)\ H = \left\{ \frac{1}{n} + (-1)^n : \ n \in \mathbb{N} \right\} \subset \mathbb{R}$

$b)\ H = \left\{ \frac{(-1)^n}{n} + 1 : n \in \mathbb{N} \right\} \subset \mathbb{R}$

$c)\ H = \left\{ \frac{1}{2^n} + \frac{1}{2^m} : m, n \in \mathbb{N} \right\} \subset \mathbb{R}$

$d)\ H = \left\{ \frac{x^2+1}{3x^2+2} : x \in \mathbb{R} \right\} \subset \mathbb{R}$

$e)\ H = \left\{ \frac{2x+3}{3x+1} : x \in \mathbb{Z} \right\} \subset \mathbb{R}$

$f)\ H = \left\{ \frac{x}{y} : 0 < x < 1,\ 0 < y < 1 \right\} \subset \mathbb{R}$

$g)\ H = \left\{ \frac{p}{q} : p, q \in \mathbb{Z},\ p^2 < q^2 \right\} \subset \mathbb{R}$

$h)\ H = \left\{ r \in Q^+ : r^2 < 2 \right\} \subset \mathbb{R}$

2. Let $\omega \in \mathbb{R}$ be a positive irrational number. Let

$$A = \{m + n\,\omega : \ m + n\,\omega > 0, \ \ m, n \in \mathbb{Z}\}$$

Prove that $\inf A = 0$.

3. Let $A \subset \mathbb{R}$ be a nonempty set that is bounded above. Show that the set $B = \{-a : a \in A\}$ is bounded below and $\inf B = -\sup A$.

4. Let $A, B \subset \mathbb{R}$ be nonempty bounded sets. Prove that
$a)\ \inf(A \cup B) = \min\{\inf A, \inf B\}$
$b)\ \sup(A \cup B) = \max\{\sup A, \sup B\}$
$c)$ if $A \cap B \neq \emptyset$ then $\inf(A \cap B) \geq \max\{\inf A, \inf B\}$
$d)$ if $A \cap B \neq \emptyset$ then $\sup(A \cap B) \leq \min\{\sup A, \sup B\}$
$e)$ if $A \subset B$ then $\inf A \geq \inf B$ and $\sup A \leq \sup B$

Complex numbers

1. Find the algebraic form $(a + i\,b)$ of the following complex numbers:

a) $\frac{3-2i}{-2+i}$ b) $\frac{2+i}{i(1-4i)}$ c) $i^2, i^3, i^4, \ldots, i^{2021}$ d) $(2+i)^{37}(2-i)^{38}$

2. Find the trigonometric form $(r\cos\varphi + i\,r\sin\varphi)$ of the following complex numbers:

a) $\sqrt{6} - i\sqrt{2}$ b) $-4i$ c) 8

3. Find the algebraic (or trigonometric) form of the following complex numbers:

a) $(1+i)^8$ b) $(1-i)^4$ c) $\left(1 + i\sqrt{3}\right)^{100}$ d) $\sqrt{i}$ e) $\sqrt[3]{1}$

f) $\sqrt[4]{-16}$ g) $\dfrac{\sqrt{i}}{1-i}$ h) $\sqrt{-5+12i}$

4. Plot the following sets on the complex plane:

a) $\{z \in \mathbb{C}: \ |z-3| \le 2\}$

b) $\{z \in \mathbb{C}: \ |z|^2 = \mathrm{Re}(z)\}$

c) $\{z \in \mathbb{C}: \ \mathrm{Re}(z+1) = |z-1|\}$

d) $\{z \in \mathbb{C}: \ |z| > 5, \ \mathrm{Im}(z) \ge \mathrm{Re}(z)\}$

e) $\{z \in \mathbb{C}: \ |z| < |z-2i|, \ |z-i| \le 1\}$

f) $\{z \in \mathbb{C}: \ |z+i| + |z+3| = 7\}$

g)* $\left\{z \in \mathbb{C}: \ \left|\dfrac{z-1}{z+i}\right| = 2\right\}$

5. Solve the following equations and equation systems on the set of complex numbers:

a) $z^6 + 16z^2 = 0$

b) $(3-i)z^2 + 3iz + 6 - i = 0$

c) $|z| - z = 1 + 2i$

d) $z^2 = \overline{z}$

e) $z^2 + (1+i)\overline{z} + 4i = 0$

f) $2iz^3 = (1+i)^8$

g) $\mathrm{Re}(z) + 2\,\mathrm{Im}(z) = 0$ and $\mathrm{Re}(z^2) - 2\,\mathrm{Im}(z) = 1$

h) $\mathrm{Re}(z^2) = 2\,\mathrm{Im}(z)$ and $\mathrm{Im}(z^2) = 2\,\mathrm{Re}(z)$

6. Assume that for the complex number z, $\mathrm{Im}(z) \neq 0$ and $\mathrm{Im}\left(z + \dfrac{1}{z}\right) = 0$. Find $|z|$.

7. For which values of n $(n \in \mathbb{N})$ is the complex number $\left(\sqrt{3} - i\right)^n$ real?

8. How many complex roots can a seventh-order real-coefficient polynomial have?

9. One vertex of a regular hexagon is $2 + i$, its center is $3 + 2i$. Find the other vertices.

10.* Prove that the sum of the vectors pointing from the center of a regular n-sided polygon to its vertices is the zero vector.

11.* Without solving the following equations, decide how many solutions they have on the set of complex numbers. Give a reason for your answer.

a) $\dfrac{1+i}{(z+i)^3} = (\overline{z} - i)^3$ b) $(\overline{z}+1)^7 = \dfrac{1}{(\overline{z})^7} - 1$

12.* It is an interesting fact that a product of two sums of squares is itself a sum of squares. For example,

$$(1^2 + 2^2)\cdot(3^2 + 4^2) = 125 = 5^2 + 10^2 = 2^2 + 11^2$$

Show that for any two pairs of integers $\{a, b\}$ and $\{c, d\}$, we can find integers u, v with

$$(a^2 + b^2)(c^2 + d^2) = u^2 + v^2$$