Calculus 1 - Topics for the Final exam

The Final exam consists of two written parts, the topics are given below.
In the oral exam (optional) any question from Part 1 and 2 can be expected.

Topics for Part 1 of the Final exam

Part 1 of the Final exam will consist of:
- definition and theorems to state, please find the topics below
- one proof of a theorem, please find a list below
- true or false questions
- examples to give (e.g. give an example of a function or a sequence with certain prescribed properties)

You are expected to be able to state the following definitions and theorems:

Number sequences

1. Convergence and divergence of sequences
2. Uniqueness of the limit
3. Convergent sequences are bounded
4. Operations with convergent sequences (limits of the sum, product, reciprocal and quotient of sequences)
5. Limit of the product of a bounded sequence and a sequence tending to zero is zero
6. The sandwich theorem for number sequences
7. Monotonic and bounded sequences are convergent
8. Subsequences, limit of subsequences of a convergent sequence
9. Bolzano-Weierstrass theorem for number sequences
10. Cauchy-sequences
11. Limit points of number sequences, limes superior, limes inferior

Numerical series

1. Definition of numerical series. Convergence, divergence, sum of a numerical series. Geometric, telescoping, harmonic series.
2. The nth term test for divergence of series
3. Convergence of the p-series (hyperharmonic series)
4. Comparison test for nonnegative series
5. Absolute and conditional convergence. Absolute convergence theorem (absolutely convergent series are convergent).
6. Leibniz rule for alternating series, error estimation for alternating series
7. Root test
8. Ratio test
9. Cauchy product of series, Mertens' theorem

Power series, Taylor polynomial, Taylor series

1. Definition of power series, Cauchy-Hadamard theorem about the radius of convergence of power series.

2. Definition of Taylor polynomials and Taylor series of a function
3. Uniqueness of the Taylor polynomial: The Taylor polynomial of order n has the same values of the first n derivatives as the function
4. Taylor's theorem with the remainder term
5. Term-by-term differentiation and integration of power series
6. Binomial series

Basic topological concepts
1. Open sets, closed sets, bounded sets
2. Interior, exterior, boundary points of a set. Limit points, isolated points of a set, closure of a set.
3. Compact sets, Cantor intersection theorem, Borel-Lebesgue theorem

Real functions
1. Definition of limits of functions, one-sided limits
2. The sequential criterion for a limit of a function
3. Operations with limits of functions (sum, difference, product and quotient of limits)
4. Sandwich theorem for limits of functions
5. Definition of continuity of a function
6. The sequential criterion for continuity
7. Algebraic properties of continuous functions (sum, difference, product and quotient of continuous functions is continuous)
8. Sandwich theorem for continuity
9. Types of discontinuities
10. Topological characterization of continuous functions
11. Intermediate value theorem or Bolzano's theorem
12. Weierstrass extreme value theorem
13. Continuous image of a compact set is compact
14. Uniform continuity, Heine's theorem, Lipschitz continuity
15. Continuity of the inverse function

Differentiation
1. Definition of the derivative of a function
2. If f is differentiable at x_0 then f is continuous at x_0
3. Operations with the derivative (sum rule, difference rule, product rule, quotient rule for the derivative)
4. The chain rule
5. Linear approximation of a function
6. Derivative of the inverse
7. Definition of a local extremum. Necessary condition for the existence of a local extremum.
8. Rolle's theorem
9. Lagrange's mean value theorem
10. Cauchy's mean value theorem
11. L'Hospital's rule
12. Local properties of the derivative
13. Darboux's theorem

14. First derivative test for monotonicity on an interval
15. Sufficient conditions for a local extremum, first derivative test and second derivative test
16. Definition of convexity and concavity
17. Necessary and sufficient conditions for convexity
18. Definition of an inflection point
19. Necessary condition for an inflection point, second derivative test
20. Sufficient conditions for an inflection point, second derivative test and third derivative test

Integration

1. Antiderivatives can differ only by a constant
2. The integration-by-parts formula
3. Partial fraction decomposition of rational functions
4. Integration by the substitution formula
5. Lower and upper Darboux sum, Riemann sum, partition and its norm
6. Definition of the Riemann integral
7. Oscillation sum, Riemann's criterion for integrability
8. Sufficient conditions for Riemann integrability
9. The first fundamental theorem of calculus (the Newton-Leibniz formula)
10. The integral function, the second fundamental theorem of calculus
11. Application of the definite integral (formulas for volume, surface and arc-length)

Proofs

You are expected to know the following proofs.

1. Sum rule and product rule for convergent sequences
2. Limit of the product of a bounded sequence and a sequence tending to zero is zero
3. The sandwich theorem for number sequences
4. Monotonic and bounded sequences are convergent (proof of the monotonically increasing case)
5. The sum of a geometric series
6. The *n*th term test for divergence of series
7. Comparison test for nonnegative series
8. Sequential criterion for a limit of a function
9. Intermediate value theorem or Bolzano's theorem
10. A differentiable function is continuous
11. The derivative of the sum, product and reciprocal
12. Necessary condition for the existence of a local extremum
13. Rolle's theorem
14. Lagrange's mean value theorem
15. Antiderivatives can differ only by a constant
16. First derivative test for monotonicity on an interval
17. Uniqueness of the Taylor polynomial: The Taylor polynomial of order *n* has the same values of the first *n* derivatives as the function
18. The Newton-Leibniz formula

Topics for Part 2 of the Final exam

Part 2 of the Final exam will consist of several exercises to solve. The exercises will be similar to the exercises of the midterm test and homeworks.

Possible topics include:

1. Logical symbols: given a statement in words, write it out with logical symbols; given a statement with logical symbols, explain what it means in words. Decide truth values.
2. Mathematical induction. Prove equalities or inequalities by induction.

3. Limits of sequences: calculate the limits of sequences (several types of sequences). Comparison of orders of magnitudes. Limit of the sequences $\left(\sqrt[n]{p}\right)$ where $p > 0$ and $\left(\sqrt[n]{n}\right)$. Subsequences, sandwich theorem. Recursively given sequences. Limit of the sequences $\left(\left(1+\frac{x}{n}\right)^n\right)$. Limit inferior, limit superior, accumulation point (limit point).

4. Convergence of series: decide whether a given series converges or diverges (applying the nth term test, ratio test, root test, comparison test, absolute convergence, Leibniz-test).
5. Sum of series: calculate the sum of a given series (geometric series, partial fraction decomposition for telescoping sums, product series).

6. Power-series: decide the radius of convergence of a given power-series.

7. Calculating limits of functions. The limit $\lim_{x\to 0}\frac{\sin x}{x}=1$. Types of discontinuities.
8. Differentiation: differentiate a given function. Differentiate implicitly given functions.
9. L’Hospital’s rule: find limits of the form "$\frac{0}{0}$", "$\frac{\infty}{\infty}$", "1^∞".

10. Analysis of functions: given a function, plot its graph after determining its properties (zeroes, limits, asymptotes, monotonically increasing and decreasing parts, local maxima and minima, convexity, inflection points).
11. Optimization problems: e.g. maximize the volume of a cylinder inscribed in a sphere.

12. Taylor polynomials: give the Taylor polynomial of a certain order of a given function around a point x_0. Calculate error bounds.

13. Indefinite integrals: finding the indefinite integral (antiderivative) of a given function. Integration by parts, and by substitution. Partial fraction decomposition, integration of rational functions. Integration of trigonometric, hyperbolic functions.

14. Definite integrals, the Newton-Leibniz formula.
15. Differentiating the integral function.
16. Calculating area, volume, surface area, arc-length.