Calculus 1, Midterm test 2

30th November, 2023

Name: ______________________________ **Neptun code: _________**

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1. (12 points) Let $H = \left\{ \left(-\frac{1}{2}\right)^n : n \in \mathbb{N}^+ \right\} \cup (2, 3) \cup ((3, \infty) \cap \mathbb{Q})$.

a) Find the set of interior points, boundary points, limit points and isolated points of H.

b) Is the set $A = \left\{ \left(-\frac{1}{2}\right)^n : n \in \mathbb{N}^+ \right\}$ closed? Why?

2. (18 points) Let $f(x) = \dfrac{|x+2| \cdot \sin(x^2 - 3x)}{x^3 + x^2 - 2x}$.

Find the points of discontinuities of f. What type of discontinuities are these?

3. (22 points) Calculate the derivatives of the following functions. Where are these functions differentiable?

a) $f(x) = \cos\left(\dfrac{2x^3 \arctan(x^2)}{2^x}\right)$ **b)** $g(x) = |x| \cdot \ln(|x| + 1)$

In case b), use the definition to calculate $g'(0)$.

4. (9+10+9 points) Calculate the following limits:

a) $\lim\limits_{x \to 0} \operatorname{ctg}(5x^3) \arctan(2x^3)$ **b)** $\lim\limits_{x \to -1+0} (x+1)^{x^2+2x+1}$ **c)** $\lim\limits_{x \to \infty} \dfrac{\cosh(3x+2)}{\cosh(5-3x)}$

5. (18 points) Analyze the following function and plot its graph: $f(x) = (x^2 - x)\, e^x$.

6. (12 points) A rectangular box with a square base and no top needs to be made using 300 square centimeters of paper. Find the lengths of the edges if the volume of the rectangular box is maximal.

7.* (10 points - BONUS) Given a regular triangle of unit side length, we cut off a corner in such a way that the removed part forms a regular triangle with side length $x \in (0, 1)$. We want to find the maximum value of the ratio $A(x)/P(x)$, where $A(x)$ is area of the remaining trapezoid and $P(x)$ is its perimeter. (Without proving it, we accept that the function under consideration has a maximum on the given interval.)

Solutions

1. (12 points) Let $H = \left\{\left(-\frac{1}{2}\right)^n : n \in \mathbb{N}^+\right\} \cup (2,\, 3) \cup ((3,\, \infty) \cap \mathbb{Q})$.

a) Find the set of interior points, boundary points, limit points and isolated points of H.

b) Is the set $A = \left\{\left(-\frac{1}{2}\right)^n : n \in \mathbb{N}^+\right\}$ closed? Why?

Solution.

a) Set of interior points: $\operatorname{int} A = (2,\ 3)$ **(2p)**

Set of boundary points: $\{0\} \cup \left\{\left(-\frac{1}{2}\right)^n : n \in \mathbb{N}^+\right\} \cup \{0\} \cup [3,\, \infty)$ **(3p)**

Set of limit points: $A' = \{0\} \cup [2,\, \infty)$ **(3p)**

Set of isolated points: $\left\{\left(-\frac{1}{2}\right)^n : n \in \mathbb{N}^+\right\}$ **(2p)**

b) The set A is not closed, since 0 is a limit of A but not included in it. **(2p)**

2. (18 points) Let $f(x) = \dfrac{|\, x+2\,| \cdot \sin(x^2 - 3x)}{x^3 + x^2 - 2x}$.

Find the points of discontinuities of f. What type of discontinuities are these?

Solution. The function $f(x) = \dfrac{|\, x+2\,| \cdot \sin(x^2 - 3x)}{x(x+2)(x-1)}$ is continuous except the points $0,\ -2,\ 1$. **(1p)**

- $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} \dfrac{\sin(x^2 - 3x)}{x^2 - 3x} \cdot \dfrac{(x-3)\cdot |\, x+2\,|}{(x+2)(x-1)} = 1 \cdot \dfrac{(-3)\cdot 2}{-2} = 3$ **(3p)**

  $\Longrightarrow$ f has a removable discontinuity at $x = 0$. **(2p)**

- $\lim\limits_{x\to -2\pm 0} f(x) = \lim\limits_{x\to -2\pm 0} \dfrac{|\, x+2\,|}{x+2} \cdot \dfrac{\sin(x^2 - 3x)}{x(x-1)} = \pm \dfrac{\sin 10}{6}$ **(4p)**

  $\Longrightarrow$ f has a jump discontinuity at $x = -2$ **(2p)**

- $\lim\limits_{x\to 1\pm 0} f(x) = \lim\limits_{x\to 1\pm 0} \dfrac{|\, x+2\,| \sin(x^2 - 3x)}{x(x+2)} \cdot \dfrac{1}{x-1} = \mp\infty$ **(4p)**

  $\Longrightarrow$ f an essential discontinuity at $x = 1$ **(2p)**

3. (22 points) Calculate the derivatives of the following functions. Where are these functions differentiable?

a) $f(x) = \cos\left(\dfrac{2x^3 \arctan(x^2)}{2^x}\right)$ **b)** $g(x) = |\, x\,| \cdot \ln(|\, x\,| + 1)$

In case b), use the definition to calculate $g'(0)$.

Solution. a) The function f is everywhere differen-
tiable.

$$f'(x) = -\sin\left(\frac{2x^3 \arctan(x^2)}{2^x}\right) \cdot \frac{\left(6x^2 \arctan(x^2) + 2x^3 \cdot \frac{2x}{x^4+1}\right) \cdot 2^x - 2x^3 \arctan(x^2) \cdot 2^x \ln 2}{4^x} \quad \textbf{(10p)}$$

b) $g'(0) = \lim\limits_{x\to 0} \dfrac{g(x) - g(0)}{x - 0} = \lim\limits_{x\to 0} \left(\dfrac{|\,x\,|}{x} \cdot \ln(\,|\,x\,|\,+1) \right) = 0$, since $\dfrac{|\,x\,|}{x} = \pm 1$ is bounded

and $\lim\limits_{x\to 0} \ln(\,|\,x\,|\,+1) = 0$.

If $x > 0$, then $g'(x) = (x \cdot \ln(x+1))' = \ln(x+1) + \dfrac{x}{x+1}$

If $x < 0$, then $g'(x) = (-x \cdot \ln(1-x))' = -\ln(1-x) + \dfrac{x}{1-x}$

The function g is everywhere differentiable. **(12p)**

4. (9+10+9 points) Calculate the following limits:

a) $\lim\limits_{x\to 0} \operatorname{ctg}(5x^3) \arctan(2x^3)$ **b)** $\lim\limits_{x\to -1+0} (x+1)^{x^2+2x+1}$ **c)** $\lim\limits_{x\to\infty} \dfrac{\cosh(3x+2)}{\cosh(5-3x)}$

Solution.

a) the limit has the type $(\pm\infty)\cdot 0$:

$$\lim_{x\to 0} \operatorname{ctg}(5x^3) \arctan(2x^3) = \lim_{x\to 0} \frac{\arctan(2x^3)}{\operatorname{tg}(5x^3)} \overset{\text{"}\frac{0}{0}\text{", L'H}}{=} \lim_{x\to 0} \frac{\frac{1}{1+4x^6} \cdot 6x^2}{\frac{1}{\cos^2(5x^3)} \cdot 15x^2} = \frac{6}{15} = \frac{2}{5} \quad \textbf{(9p)}$$

b) the limit has the type 0^0:

$$\lim_{x\to -1+0} (x+1)^{x^2+2x+1} = \lim_{x\to -1+0} \left(e^{\ln(x+1)}\right)^{x^2+2x+1} = e^{\lim_{x\to -1+0}\left(x^2+2x+1\right)\ln(x+1)} \quad \textbf{(5p)}$$

and $\lim\limits_{x\to -1+0} \left(x^2+2x+1\right)\ln(x+1) = \lim\limits_{x\to -1+0} \dfrac{\ln(x+1)}{\frac{1}{(x+1)^2}} \overset{\text{"}\frac{-\infty}{\infty}\text{", L'H}}{=} \lim\limits_{x\to -1+0} \dfrac{\frac{1}{x+1}}{\frac{-2}{(x+1)^3}} = 0$ **(4p)**

so the limit is $e^0 = 1$. **(1p)**

c) $$\lim_{x\to\infty} \frac{\cosh(3x+2)}{\cosh(5-3x)} = \lim_{x\to\infty} \frac{e^{3x+2} + e^{-3x-2}}{e^{5-3x} + e^{3x-5}} = \lim_{x\to\infty} \frac{e^{3x}}{e^{3x}} \cdot \frac{e^2 + e^{-6x-2}}{e^{5-6x} + e^{-5}} = \frac{e^2 + 0}{0 + e^{-5}} = e^7 \quad \textbf{(9p)}$$

5. (18 points) Analyze the following function and plot its graph: $f(x) = \left(x^2 - x\right) e^x$.

Solution.

$D_f = \mathbb{R}$; $f(x) = 0 \iff x = 0$ or $x = 1$; $\lim\limits_{x\to +\infty} f(x) = \infty$, $\lim\limits_{x\to -\infty} f(x) = 0$

$f'(x) = e^x \left(x^2 + x - 1\right) = 0 \iff x = x_1 = \frac{1}{2}\left(-1 - \sqrt{5}\right)$ or $x = x_2 = \frac{1}{2}\left(-1 + \sqrt{5}\right)$ **(3p)**

x	x<x₁	x=x₁	x₁<x<x₂	x=x₂	x>x₂
f'	+	0	−	0	+
f	↗	loc. max	↘	loc. min	↗

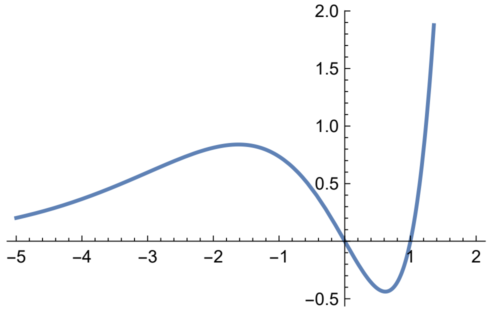
(4p)

$(x_1 \approx -1.61803, \quad x_2 \approx 0.618034, \quad f(x_1) \approx 0.839962, \quad f(x_2) \approx -0.437971)$

$f''(x) = e^x \cdot x(x+3) = 0 \iff x_1 = -3,\ x_2 = 0$ **(3p)**

x	x<-3	x=-3	-3<x<0	x=0	0<x
f''	+	0	-	0	+
f	$\cup$	infl:0	$\cap$	infl: $\frac{12}{e^3}$	$\cup$

(4p)



(3p)

6. (12 points) A rectangular box with a square base and no top needs to be made using 300 square centimeters of paper. Find the lengths of the edges if the volume of the rectangular box is maximal.

Solution. The surface area of the box with base x and height y is

$A = x^2 + 4\,x\,y = 300 \implies y = \dfrac{300 - x^2}{4\,x}$

The volume of the box is $V = x^2\,y$

We want to find the maximum of the function $V(x) = x^2 \cdot \dfrac{300 - x^2}{4\,x} = \dfrac{1}{4}\left(300\,x - x^3\right)$. **(5p)**

$V'(x) = \dfrac{1}{4}\left(300 - 3\,x^2\right) = 0 \implies x = 10$, since $x > 0$.

$V''(x) = \dfrac{1}{4} \cdot (-6\,x) \implies V''(10) = \dfrac{1}{4} \cdot (-60) < 0 \implies V$ has a local maximum at $x = 10$. **(5p)**

The side lengths of the box are: $x = 10$ cm, $y = 3$ cm. **(2p)**

7.* (10 points - BONUS) Given a regular triangle of unit side length, we cut off a corner in such a way that the removed part forms a regular triangle with side length $x \in (0,\ 1)$. We want to find the maximum value of the ratio $A(x)/P(x)$, where $A(x)$ is area of the remaining trapezoid and $P(x)$ is its perimeter. (Without proving it, we accept that the function under consideration has a maximum on the given interval.)

Solution. $A(x) = \dfrac{\sqrt{3}}{4}\left(1 - x^2\right)$ **(2p)**, $P(x) = 3 - x$ **(1p)**, so

$\left(\dfrac{A}{P}\right)'(x) = \dfrac{\sqrt{3}}{4} \cdot \dfrac{-2\,x(3 - x) + \left(1 - x^2\right)}{(3 - x)^2}$ **(4p)**

This function is differentiable on the interval $(0,\ 1)$, therefore it may have a local extremum only at the zeros of the derivative. The denominator is positive, the roots of the numerator are $3 \pm \sqrt{8}$ **(2p)**. Out of the two numbers, only the root $3 - \sqrt{8}$ is in the interval $(0,\ 1)$. **(1p)**