Calculus 1, Midterm Test 2

1st December, 2022

Name: ________________________________ **Neptun code:** __________

1.	2.	3.	4.	5.	6.	7.	$\sum$

1. (10 points) Find the interval of convergence of the following power series:

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{(n+3)\cdot 2^n}(x+5)^n$$

2. (12 points) Let $A = ([0,2]\setminus\mathbb{Q}) \cup (4,5) \cup (5,6]$.

a) Find the set of interior points, boundary points, limit points and isolated points of A.

b) Find the closure of A.

3. (18 points) Determine the points of discontinuity of the following function. What type of discontinuities are these?

$$f(x) = \arctan\left(\frac{1}{x+2}\right) + \frac{x^2-1}{x^2+2x-3}$$

4. (10 points) Find the values of the parameters such that the following function be differentiable on $\mathbb{R}$:

$$f(x) = \begin{cases} \dfrac{x^2}{x+1} & \text{if } x \geq 1 \\ a x^2 + b & \text{if } x < 1 \end{cases}$$

5. (10 points) Find the equation of the tangent line to the function $f(x) = \dfrac{\cos(2x) + \ln(x+1)}{\sqrt{x^2+1}}$ at $x_0 = 0$.

6. (10+10+10+10 points) Calculate the following limits:

a) $\displaystyle\lim_{x\to-\infty}\left(\sqrt{x^2+x} - \sqrt{x^2+5x+3}\right)$

b) $\displaystyle\lim_{x\to 0}\left(\frac{1}{e^{2x}-1} - \frac{1}{2x}\right)$

c) $\displaystyle\lim_{x\to 0}\frac{\sin(4x^2)}{\ln(\cos(2x))}$

d) $\displaystyle\lim_{x\to\infty}\frac{e^x\cosh(2x)}{\sinh(3x)}$

7.* (10 points - BONUS) Is it true that if $x > 0$ then $x < (1+x)\ln(1+x) < x(1+x)$?

(Help: Investigate the function $f(x) = \ln x$ on the interval $[1, 1+x]$.)

Solutions

1. (10 points) Find the interval of convergence of the following power series:

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{(n+3)\cdot 2^n} (x+5)^n$$

Solution. The coefficients are $a_n = \dfrac{(-1)^n}{(n+3)\cdot 2^n}$ and the center is

$x_0 = -5$.

$$\sqrt[n]{|\,a_n\,|} = \sqrt[n]{\left|\frac{(-1)^n}{(n+3)\cdot 2^n}\right|} = \frac{1}{\sqrt[n]{n+3}\cdot 2} \longrightarrow \frac{1}{1\cdot 2} = \frac{1}{2} = \frac{1}{R} \implies R = 2. \ \textbf{(3p)}$$

$\sqrt[n]{n+3} \longrightarrow 1$ by the sandwich theorem, since $1 \le \sqrt[n]{n+3} \le \sqrt[n]{n+3n} = \sqrt[n]{4}\cdot\sqrt[n]{n} \longrightarrow 1\cdot 1 = 1$.

Let H denote the domain of convergence. Then $(-7, -3) \subset H \subset [-7, -3]$.

The endpoints of H:

If $x = x_0 + R = -3$ then the series is $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{(n+3)\cdot 2^n} 2^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n+3}$.

This is a Leibniz series, so it is convergent. $\implies -3 \in H$.

If $x = x_0 - R = -7$ then the series is $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{(n+3)\cdot 2^n} (-2)^n = \sum_{n=1}^{\infty} \frac{1}{n+3}$

Since $\dfrac{1}{n+3} \ge \dfrac{1}{n+3n} = \dfrac{1}{4n}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4n}$ diverges, then by the comparison test, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+3}$ also diverges. $\implies$

$-7 \notin H$. **(2p)**

The domain of convergence is $H = (-7, -3]$. **(2p)**

2. (12 points) Let $A = ([0, 2] \setminus \mathbb{Q}) \cup (4, 5) \cup (5, 6]$.

a) Find the set of interior points, boundary points, limit points and isolated points of A.

b) Find the closure of A.

Solution:

a)

Set of interior points:	$\text{int}\, A = (4, 5) \cup (5, 6)$ **(2p)**
Set of boundary points:	$\partial A = [0, 2] \cup \{4, 5, 6\}$ **(3p)**
Set of limit points:	$A' = [0, 2] \cup [4, 6]$ **(3p)**
Set of isolated points:	$\emptyset$ **(2p)**

b)

The closure of A:	$\overline{A} = [0, 2] \cup [4, 6]$ **(2p)**

3. (18 points) Determine the points of discontinuity of the following function. What type of discontinuities are these?

$$f(x) = \arctan\left(\frac{1}{x+2}\right) + \frac{x^2-1}{x^2+2x-3}$$

Solution. $f(x) = \arctan\left(\frac{1}{x+2}\right) + \frac{(x-1)(x+1)}{(x-1)(x+3)}$

Since the arctan function and the polynomials are continuous and the composition and ratio of continuous functions is continuous if the denominator is not 0, then f is continuous on its domain. The points of discontinuities are $x_1 = -2, x_2 = 1, x_3 = -3$. **(3p)**

a) If $x_1 = -2$: $\lim\limits_{x\to -2+0} \frac{1}{x+2} = \frac{1}{0+} = +\infty \Longrightarrow \lim\limits_{x\to -2+} \arctan\left(\frac{1}{x+2}\right) = \frac{\pi}{2}$

$\lim\limits_{x\to -2-0} \frac{1}{x+2} = \frac{1}{0-} = -\infty \Longrightarrow \lim\limits_{x\to -2-} \arctan\left(\frac{1}{x+2}\right) = -\frac{\pi}{2}$ **(2p)**

$\Longrightarrow \lim\limits_{x\to -2\pm 0} f(x) = \pm\frac{\pi}{2} + \lim\limits_{x\to -2\pm 0} \frac{(x-1)(x+1)}{(x-1)(x+3)} = \pm\frac{\pi}{2} - 1$ **(1p)**

$\Longrightarrow$ f has a jump discontinuity at $x_1 = -2$. **(2p)**

b) If $x_2 = 1$: $\lim\limits_{x\to 1\pm 0} f(x) = \arctan\frac{1}{3} + \lim\limits_{x\to 1\pm 0} \frac{(x-1)(x+1)}{(x-1)(x+3)} = \arctan\frac{1}{3} + \frac{1}{2}$ **(3p)**

$\Longrightarrow$ f has a removable discontinuity at $x_2 = 1$. **(2p)**

c) If $x_3 = -3$: $\lim\limits_{x\to -3\pm 0} f(x) = \arctan(-1) + \lim\limits_{x\to -3\pm 0} \frac{(x-1)(x+1)}{(x-1)(x+3)} = -\frac{\pi}{4} + (-2) \lim\limits_{x\to -3\pm 0} \frac{1}{x+3} =$

$= -\frac{\pi}{4} + (-2)\cdot(\pm\infty) = \mp\infty$ **(3p)**

$\Longrightarrow$ f has an essential discontinuity at $x_3 = -3$. **(2p)**

4. (10 points) Find the values of the parameters such that the following function be differentiable on $\mathbb{R}$: $f(x) = \begin{cases} \frac{x^2}{x+1} & \text{if } x \geq 1 \\ a x^2 + b & \text{if } x < 1 \end{cases}$

Solution. The function is differentiable for all a, b except $x = 1$.

If f is continuous at $x = 1$ then $\lim\limits_{x\to 1+0} f(x) = \lim\limits_{x\to 1-0} f(x) = f(1) \Longrightarrow \frac{1}{2} = a + b$ **(3p)**

$f'(x) = \begin{cases} \frac{2x\cdot(x+1) - x^2}{(x+1)^2} = \frac{x^2 + 2x}{(x+1)^2} & \text{if } x > 1 \\ 2ax & \text{if } x < 1 \end{cases}$ **(2p)**

If f is differentiable at $x = 1$ then $\lim\limits_{x\to 1+0} f'(x) = \lim\limits_{x\to 1-0} f'(x) \Longrightarrow \frac{3}{4} = 2a$ **(3p)**.

The solution of the equation system is $a = \frac{3}{8}$, $b = \frac{1}{8}$. **(1p)**

5. (10 points) Find the equation of the tangent line to the function $f(x) = \frac{\cos(2x) + \ln(x+1)}{\sqrt{x^2+1}}$ at $x_0 = 0$.

Solution. $f'(x) = \frac{1}{x^2+1}\left(\left(-2\sin(2x) + \frac{1}{x+1}\right)\cdot\sqrt{x^2+1} - (\cos(2x) + \ln(x+1))\cdot\frac{1}{2}(x^2+1)^{-\frac{1}{2}}\cdot 2x\right)$ **(5p)**

$f(0) = 1,\ \ f'(0) = 1$ **(1p)**

The equation of the tangent line is $y = f(0) + f'(0)(x-0)$, that is, $y = 1 + x$ **(3p)**

6. (10+10+10+10 points) Calculate the following limits:

a) $\lim\limits_{x\to-\infty}\left(\sqrt{x^2+x} - \sqrt{x^2+5x+3}\right)$ b) $\lim\limits_{x\to 0}\left(\frac{1}{e^{2x}-1} - \frac{1}{2x}\right)$

c) $\lim\limits_{x\to 0}\frac{\sin(4x^2)}{\ln(\cos(2x))}$ d) $\lim\limits_{x\to\infty}\frac{e^x\cosh(2x)}{\sinh(3x)}$

Solution.

a) $\lim\limits_{x\to-\infty}\left(\sqrt{x^2+x} - \sqrt{x^2+5x+3}\right) = \lim\limits_{x\to-\infty}\left(\sqrt{x^2+x} - \sqrt{x^2+5x+3}\right)\cdot\frac{\sqrt{x^2+x} + \sqrt{x^2+5x+3}}{\sqrt{x^2+x} + \sqrt{x^2+5x+3}}$

$$= \lim_{x\to-\infty}\frac{(x^2+x) - (x^2+5x+3)}{\sqrt{x^2+x} + \sqrt{x^2+5x+3}} = \lim_{x\to-\infty}\frac{-4x-3}{\sqrt{x^2+x} + \sqrt{x^2+5x+3}} \quad \textbf{(6p)}$$

$$= \lim_{x\to-\infty}\frac{x}{\sqrt{x^2}}\ \frac{-4-\frac{3}{x}}{\sqrt{1+\frac{1}{x}} + \sqrt{1+\frac{5}{x}+\frac{3}{x^2}}} = (-1)\cdot\frac{-4-0}{\sqrt{1+0} + \sqrt{1+0+0}} = 2 \quad \textbf{(4p)}$$

Here $\frac{x}{\sqrt{x^2}} = \frac{x}{|x|} = \frac{x}{-x} = -1$, since $x < 0$.

b) $\lim\limits_{x\to 0}\left(\frac{1}{e^{2x}-1} - \frac{1}{2x}\right) = \lim\limits_{x\to 0}\frac{2x - (e^{2x}-1)}{2x(e^{2x}-1)}$

The limit has the form $\frac{0}{0}$ $\Longrightarrow$ L'Hospital's rule can be applied:

$$\stackrel{L'H}{=} \lim_{x\to 0}\frac{2-2e^{2x}}{2(e^{2x}-1) + 2x\cdot e^{2x}\cdot 2} \ \textbf{(4p)} \ \stackrel{L'H}{=} \lim_{x\to 0}\frac{-4e^{2x}}{4e^{2x} + 4\cdot e^{2x} + 4x\cdot e^{2x}\cdot 2} \ \textbf{(4p)} \ = \frac{-4}{8} = -\frac{1}{2} \ \textbf{(2p)}$$

c) The limit has the form $\frac{0}{0}$ $\Longrightarrow$ L'Hospital's rule can be applied:

$$\lim_{x\to 0}\frac{\sin(4x^2)}{\ln(\cos(2x))} \stackrel{L'H}{=} \lim_{x\to 0}\frac{\cos(4x^2)\cdot 8x}{\frac{1}{\cos(2x)}\cdot(-\sin(2x))\cdot 2} \quad \textbf{(5p)}$$

$$= \lim_{x\to 0}\cos(2x)\cdot\cos(4x^2)\frac{2x}{\sin(2x)}\cdot(-2) = 1\cdot 1\cdot 1\cdot(-2) = -2 \quad \textbf{(5p)}$$

d) By the definition of the functions:

$$\lim_{x\to\infty}\frac{e^x\cosh(2x)}{\sinh(3x)} = \lim_{x\to\infty}\frac{e^x(e^{2x}+e^{-2x})}{e^{3x}-e^{-3x}} = \lim_{x\to\infty}\frac{e^{3x}+e^{-x}}{e^{3x}-e^{-3x}} \ \textbf{(4p)} \ = \lim_{x\to\infty}\frac{e^{3x}}{e^{3x}}\ \frac{1-e^{-4x}}{1+e^{-6x}} \ \textbf{(3p)} \ = \frac{1-0}{1+0} = 1 \ \textbf{(3p)}$$

7.* (10 points - BONUS) Is it true that if $x > 0$ then $x < (1 + x)\ln(1 + x) < x(1 + x)$?
(Help: Investigate the function $f(x) = \ln x$ on the interval $[1, 1 + x]$.)

Solution. Let $x > 0$, then the function $f(x) = \ln x$ is continuous on $[1, 1 + x]$ and differentiable on $(1, 1 + x)$, so by Lagrange's theorem there exists $c \in (1, 1 + x)$ such that

$$\frac{\ln(1 + x) - \ln 1}{(1 + x) - 1} = \frac{\ln(1 + x)}{x} = \ln'(c) = \frac{1}{c}$$

Since $1 < c < 1 + x$ then $\frac{1}{1 + x} < \frac{1}{c} < 1 \Longrightarrow \frac{1}{1 + x} < \frac{\ln(1 + x)}{x} < 1$.
Multiplying by $x(1 + x) > 0$ we get that $x < (1 + x)\ln(1 + x) < x(1 + x)$. **(10 points)**