Calculus 1, Midterm Test 1

27th October, 2022

Name: ___________________________________ Neptun code: __________

1.	2.	3.	4.	5.	6.	7.	8.	9.	$\sum$

1. (10 points) Find those solutions z of the following equation for which $\mathrm{Re}(z) > 0$ and $\mathrm{Im}(z) < 0$. Give these solutions in algebraic form.

$$z^6 + 7z^3 - 8 = 0$$

2. (9 points) Let $a_n = \dfrac{3n^4 - 5n}{n^4 + n + 2}$. Find the limit of a_n and provide a threshold index N for $\varepsilon = 0.001$.

3. (9 points) Find the limit of the following sequence: $a_n = n\left(\sqrt{n^4 + 8n} - \sqrt{n^4 - 1}\right)$.

4. (6+6+6 points) Find the limit of the following sequences:

a) $a_n = \left(\dfrac{n^2+1}{n^2+4}\right)^{n^2}$ b) $b_n = \left(\dfrac{n^2+1}{n^2+4}\right)^{n}$ c) $c_n = \left(\dfrac{n^2+1}{n^2+4}\right)^{n^3}$

5. (12 points) Let $a_1 = 3$ and $a_{n+1} = \dfrac{10}{7 - a_n}$ for all $n \in \mathbb{N}$. Prove that (a_n) is convergent and calculate its limit.

6. (9 points) Find the liminf and limsup of the following sequence. Is this sequence convergent?

$$a_n = \sqrt[n]{\frac{n^4 + (-1)^n \cdot n^4}{6n^2 - n + 5}}.$$

7. (6 points) Calculate the sum of the following series: $\displaystyle\sum_{n=2}^{\infty} \frac{2^{2n+1} + 5\cdot(-2)^n}{2^{3n}}$

8. (9+9+9 points) Decide whether the following series are convergent or divergent:

a) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n^4 + 3n} + 1}{n^6 + n + \ln(n)}$ b) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{(2n)!}}{(n+3)!}$ c) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n+3}{2n+3}\right)^{n^2}$

9.* (10 points - BONUS): Construct a number sequence whose limit points are the positive integers. Give a reason for your answer.

Solutions

1. (10 points) Find those solutions z of the following equation for which $\text{Re}(z) > 0$ and $\text{Im}(z) < 0$. Give these solutions in algebraic form.

$$z^6 + 7z^3 - 8 = 0$$

Solution. $z^6 + 7z^3 - 8 = (z^3 + 8)(z^3 - 1) = 0 \iff z^3 = -8$ or $z^3 = 1$. **(1p)**

a) If $z^3 = -8 = 8(\cos\pi + i\sin\pi)$ then $z_k = 2\left(\cos\frac{\pi + k\cdot 2\pi}{3} + i\sin\frac{\pi + k\cdot 2\pi}{3}\right)$, where $k = 0, 1, 2$. **(2p)**

$z_0 = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = 1 + \sqrt{3}\,i$

$z_1 = 2(\cos\pi + i\sin\pi) = -2$

$z_3 = 2\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right) = 1 - \sqrt{3}\,i$

From here the condition $\text{Re}(z) > 0$, $\text{Im}(z) < 0$ holds for $1 - \sqrt{3}\,i$. **(3p)**

b) If $z^3 = 1 = (\cos 0 + i\sin 0)$ then $z_k = \cos\frac{k\cdot 2\pi}{3} + i\sin\frac{k\cdot 2\pi}{3}$, where $k = 0, 1, 2$. **(2p)**

$z_0 = \cos 0 + i\sin 0 = 1$

$z_1 = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$

$z_3 = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$

From here no solutions are suitable. **(2p)**

2. (9 points) Let $a_n = \frac{3n^4 - 5n}{n^4 + n + 2}$. Find the limit of a_n and provide a threshold index N for $\varepsilon = 0.001$.

Solution. $a_n = \frac{3n^4 - 5n}{n^4 + n + 2} = \frac{3 - \frac{5}{n^3}}{1 + \frac{1}{n^3} + \frac{2}{n^4}} \longrightarrow \frac{3 - 0}{1 + 0 + 0} = 3$ **(1p)**

Let $\varepsilon > 0$. We have to find $N(\varepsilon) \in \mathbb{N}$ such that if $n > N$ then $|a_n - A| < \varepsilon$. $(A = 3)$ **(1p)**

$$|a_n - A| = \left|\frac{3n^4 - 5n}{n^4 + n + 2} - 3\right| = \left|\frac{3n^4 - 5n - 3\cdot(n^4 + n + 2)}{n^4 + n + 2}\right| =$$

$$= \left|\frac{-8n - 6}{n^4 + n + 2}\right| = \frac{8n + 6}{n^4 + n + 2} \ (2p) \le \frac{8n + 6n}{n^4 + 0 + 0} = \frac{14}{n^3} < \varepsilon \iff n > \sqrt[3]{\frac{14}{\varepsilon}}, \ \textbf{(2p)}$$

so with the choice $N(\varepsilon) \ge \left[\frac{14}{\varepsilon}\right]$ the definition holds. **(2p)**

If $\varepsilon = 0.001$ then $N \ge 14000$. **(1p)**

3. (9 points) Find the limit of the following sequence: $a_n = n\left(\sqrt{n^4 + 8n} - \sqrt{n^4 - 1}\right)$.

Solution. $a_n = n\left(\sqrt{n^4+8n} - \sqrt{n^4-1}\right) \cdot \dfrac{\sqrt{n^4+8n} + \sqrt{n^4-1}}{\sqrt{n^4+8n} + \sqrt{n^4-1}} =$ **(2p)**

$$= n \cdot \frac{(n^4+8n) - (n^4-1)}{\sqrt{n^4+8n} + \sqrt{n^4-1}} = n \cdot \frac{8n+1}{\sqrt{n^4+8n} + \sqrt{n^4-1}} =$$

$$= \frac{n^2}{n^2} \cdot \frac{8 + \frac{1}{n}}{\sqrt{1 + \frac{8}{n^3}} + \sqrt{1 - \frac{1}{n^4}}} \textbf{ (4p) } \longrightarrow \frac{8+0}{\sqrt{1+0} + \sqrt{1-0}} = 4 \textbf{ (3p)}$$

4. (6+6+6 points) Find the limit of the following sequences:

a) $a_n = \left(\dfrac{n^2+1}{n^2+4}\right)^{n^2}$ b) $b_n = \left(\dfrac{n^2+1}{n^2+4}\right)^{n}$ c) $c_n = \left(\dfrac{n^2+1}{n^2+4}\right)^{n^3}$

Solution. a) $a_n = \left(\dfrac{n^2+1}{n^2+4}\right)^{n^2} = \dfrac{\left(1 + \frac{1}{n^2}\right)^{n^2}}{\left(1 + \frac{4}{n^2}\right)^{n^2}} \longrightarrow \dfrac{e}{e^4} = \dfrac{1}{e^3}$ **(3+2+1p)**

b) $b_n = \sqrt[n]{a_n}$ **(1p)**. Since $a_n \longrightarrow \dfrac{1}{e^3}$ and $0 < \dfrac{1}{e^3} < 1$ then there exists $N \in \mathbb{N}$ such that if $n > N$ then $\dfrac{1}{2e^3} < a_n < 1 \implies \sqrt[n]{\dfrac{1}{2e^3}} < b_n < 1$ **(3p)**. Since $\sqrt[n]{\dfrac{1}{2e^3}} \longrightarrow 1$ then by the sandwich theorem $b_n \longrightarrow 1$. **(2p)**

c) $c_n = a_n^n$ **(1p)**. Let $\dfrac{1}{e^3} < q < 1$. Since $a_n \longrightarrow \dfrac{1}{e^3}$ then there exists $N \in \mathbb{N}$ such that if $n > N$ then $0 < a_n < q \implies 0 < c_n < q^n$ **(3p)**. Since $q^n \longrightarrow 0$ then by the sandwich theorem $c_n \longrightarrow 0$ **(2p)**.

5. (12 points) Let $a_1 = 3$ and $a_{n+1} = \dfrac{10}{7 - a_n}$ for all $n \in \mathbb{N}$. Prove that (a_n) is convergent and calculate its limit.

Solution. If $\exists \lim\limits_{n\to\infty} a_n = A$ then $A = \dfrac{10}{7-A} \iff A(7-A) - 10 = 0 \iff A^2 - 7A + 10 = (A-2)(A-5) = 0$
$\iff A_1 = 2,\ A_2 = 5$ **(3p)**.

Boundedness: we prove by induction that $2 < a_n < 5$ for all $n \in \mathbb{N}$.

(1) $2 < a_1 = 3 < 5$

(2) Assume that $2 < a_n < 5$

(3) Then $-2 > -a_n > -5 \implies 5 > 7 - a_n > 2 \implies \dfrac{1}{5} < \dfrac{1}{7-a_n} < \dfrac{1}{2} \implies 2 < a_{n+1} = \dfrac{10}{7-a_n} < 5$

So (a_n) is bounded above. **(3p)**

Monotonicity: we prove by induction that (a_n) is monotonically decreasing, that is, $a_n > a_{n+1}$ for all $n \in \mathbb{N}$.

(1) $a_1 = 3 > a_2 = \frac{10}{7-3} = \frac{10}{4} = 2.5$

(2) Assume that $a_n > a_{n+1}$

(3) Then $-a_n < -a_{n+1} \implies 7 - a_n < 7 - a_{n+1}$. Since $2 < a_n < 5$ then $7 - a_n > 0$

$\implies a_{n+1} = \frac{10}{7-a_n} > \frac{10}{7-a_{n+1}} = a_{n+2}$

So (a_n) is monotonically decreasing. **(3p)**

Since (a_n) is monotonically decreasing and bounded below then it is convergent.
Since $a_1 = 3$ and the sequence is monotonically decreasing then $A = 5$ cannot be the limit.
So $\lim_{n\to\infty} a_n = 2$. **(3p)**

6. (9 points) Find the liminf and limsup of the following sequence. Is this sequence convergent?

$$a_n = \sqrt[n]{\frac{n^4 + (-1)^n \cdot n^4}{6n^2 - n + 5}}.$$

Solution. If n is odd then $a_n = 0$ **(1p)**

If n is even then $a_n = \sqrt[n]{\frac{2n^4}{6n^2 - n + 5}}$ **(1p)**

Upper estimation:

$$a_n = \sqrt[n]{\frac{2n^4}{6n^2 - n + 5}} \leq \sqrt[n]{\frac{2n^4}{6n^2 - n^2 + 0}} = \sqrt[n]{\frac{2n^2}{5}} = \sqrt[n]{\frac{2}{5}} \cdot \left(\sqrt[n]{n}\right)^2 \longrightarrow 1 \cdot 1^2 = 1 \text{ (2p)}$$

Lower estimation:

$$a_n = \sqrt[n]{\frac{2n^4}{6n^2 - n + 5}} \geq \sqrt[n]{\frac{2n^4}{6n^2 + 0 + 5n^2}} = \sqrt[n]{\frac{2n^2}{11}} = \sqrt[n]{\frac{2}{11}} \cdot \left(\sqrt[n]{n}\right)^2 \longrightarrow 1 \cdot 1^2 = 1 \text{ (2p)}$$

By the sandwich theorem $a_{2n} \longrightarrow 1$. **(1p)**

The limit points of the sequence are 0 and 1, so $\liminf a_n = 0$ and $\limsup a_n = 1$ **(1p)**
Since these are not equal then a_n is not convergent. **(1p)**

7. (6 points) Calculate the sum of the following series: $\sum_{n=2}^{\infty} \frac{2^{2n+1} + 5 \cdot (-2)^n}{2^{3n}}$

Solution. $\sum_{n=2}^{\infty} \frac{2^{2n+1} + 5 \cdot (-2)^n}{2^{3n}} = \sum_{n=2}^{\infty} \frac{2 \cdot 4^n + 5 \cdot (-2)^n}{8^n} = \sum_{n=2}^{\infty} \left(2 \cdot \left(\frac{4}{8}\right)^n + 5 \cdot \left(-\frac{2}{8}\right)^n\right) =$ **(2p)**

$= 2 \cdot \frac{\left(\frac{1}{2}\right)^2}{1 - \frac{1}{2}} + 5 \cdot \frac{\left(-\frac{1}{4}\right)^2}{1 - \left(-\frac{1}{4}\right)}$ **(4p)** $= 1 + \frac{1}{4} = \frac{5}{4}$

8. (9+9+9 points) Decide whether the following series are convergent or divergent:

a) $\sum_{n=1}^{\infty} \frac{\sqrt{n^4+3n}+1}{n^6+n+\ln(n)}$ b) $\sum_{n=1}^{\infty} \frac{\sqrt{(2n)!}}{(n+3)!}$ c) $\sum_{n=1}^{\infty} \left(\frac{n+3}{2n+3}\right)^{n^2}$

Solution. a) $0 < a_n = \frac{\sqrt{n^4+3n}+1}{n^6+n+\ln(n)} \leq \frac{\sqrt{n^4+3n^4}+1}{n^6+0+0} \leq \frac{2n^2+n^2}{n^6+0+0} = \frac{3}{n^4}$ **(6p)**

and $\sum_{n=1}^{\infty} \frac{3}{n^4}$ is convergent, so by the comparison test, the series $\sum_{n=1}^{\infty} a_n$ is convergent. **(3p)**

b) Let $a_n = \frac{\sqrt{(2n)!}}{(n+3)!}$. By the ratio test:

$$\frac{a_{n+1}}{a_n} = \frac{\sqrt{(2n+2)!}}{(n+4)!} \cdot \frac{(n+3)!}{\sqrt{(2n)!}} \textbf{ (3p) } = \frac{\sqrt{(2n+2)(2n+1)}}{n+4} \textbf{ (2p) } = \frac{\sqrt{4n^2+6n+2}}{n+4} =$$

$$= \frac{n}{n} \cdot \frac{\sqrt{4+\frac{6}{n}+\frac{2}{n^2}}}{1+\frac{4}{n}} \longrightarrow \frac{\sqrt{4+0+0}}{1+0} = 2 \textbf{ (3p) } > 1 \implies \text{the series } \sum_{n=1}^{\infty} a_n \text{ is divergent } \textbf{(1p)}$$

c) Let $a_n = \left(\frac{n+3}{2n+3}\right)^{n^2}$. By the root test:

$$\sqrt[n]{a_n} = \left(\frac{n+3}{2n+3}\right)^n \textbf{ (3p) } = \left(\frac{n}{2n} \cdot \frac{1+\frac{3}{n}}{1+\frac{3}{2n}}\right)^n = \left(\frac{1}{2}\right)^n \cdot \frac{\left(1+\frac{3}{n}\right)^n}{\left(1+\frac{3}{2n}\right)^n} \textbf{ (3p) } \longrightarrow 0 \cdot \frac{e^3}{e^{\frac{3}{2}}} = 0 \textbf{ (2p) } < 1$$

$\implies$ the series $\sum_{n=1}^{\infty} a_n$ is convergent **(1p)**

9.* (10 points - BONUS): Construct a number sequence whose limit points are the positive integers. Give a reason for your answer.

Solution. Let the sequence by the following:

1, 1, 2, 1, 2, 3, 1, 2, 3, 4, 1, 2, 3, 4, 5, 1, 2, 3, 4, 5, 6, ... **(6p)**

It can be seen that every positive integer occurs infinitely many times in this sequence and no other number occurs, so this sequence satisfies the conditions. **(4p)**

Remark: The solution is worth 2 points if for every positive integer *n* a sequence converging to *n* is given. The solution is worth 4-6 points if there are good experiments for combining infinitely many convergent sequences but the solution is not perfect or no reason is given.