Calculus 1, Repeated midterm test 2

11th December, 2023

Name: ___________________________________ Neptun code: __________

1.	2.	3.	4.	5.	6.	7.	$\sum$

1. (12 points) Let $H = ([-1, 1] \setminus \mathbb{Q}) \cup [2, 3) \cup (3, 4)$.

a) Find the set of interior points, boundary points, limit points and isolated points of *H*.

b) Is the set $A = [-1, 1] \setminus \mathbb{Q}$ closed? Why?

2. (18 points) Let $f(x) = \dfrac{(x+2)\cdot \sin^2(x-1)}{|\,x^3 + x^2 - 2x\,|}$.

Find the points of discontinuities of *f*. What type of discontinuities are these?

3. (25 points) Let $f(x) = \sqrt[3]{x(x+2)} \cdot \sin\left(\frac{\pi}{4} x\right)$.

Investigate whether the following tangent lines of *f* exist. If so, give their equations.

a) The tangent line at $x_0 = 2$.

b) The tangent line at $x_0 = 0$.

In case b), use the definition to calculate $f'(0)$.

4. (10+5+10 points) Calculate the following limits:

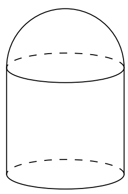
a) $\lim\limits_{x\to 0} \dfrac{e^{-x} - 1}{\arctan x}$ **b)** $\lim\limits_{x\to \infty} \dfrac{e^{-x} - 1}{\arctan x}$ **c)** $\lim\limits_{x\to 1+0} \left(\ln(x-1)\cdot \cot\left(\frac{\pi}{2} x\right)\right)$

5. (18 points) Analyze the following function and plot its graph: $f(x) = \dfrac{(x+2)^2}{x^2}$.

6. (12 points) We want to make a box with an open top from a 1 m by 1 m piece of cardboard, such that we cut out congruent squares from its corners, and turn up the sides. What are the sides of the box with maximal volume? What is this maximal volume?

7.* (10 points - BONUS)

We would like to pour oil into a bottle that is cylindrical at the bottom and hemispherical at the top, for which we have plastic for a total surface area of 30 dm^2. Determine the radius of the base circle of the cylinder so that as much oil as possible fits into the bottle.



Solutions

1. (12 points) Let $H = ([-1, 1] \setminus \mathbb{Q}) \cup [2, 3) \cup (3, 4)$.

a) Find the set of interior points, boundary points, limit points and isolated points of H.

b) Is the set $A = [-1, 1] \setminus \mathbb{Q}$ closed? Why?

Solution.

a) Set of interior points: $\text{int} A = (2, 3) \cup (3, 4)$ **(2p)**

Set of boundary points: $[-1, 1] \cup \{2, 3, 4\}$ **(3p)**

Set of limit points: $A' = [-1, 1] \cup [2, 4]$ **(3p)**

Set of isolated points: there are no isolated points **(2p)**

b) The set A is not closed, since the irrational numbers in $[0, 1]$ are limit points of A but not included in it. **(2p)**

2. (18 points) Let $f(x) = \dfrac{(x+2) \cdot \sin^2(x-1)}{|x^3 + x^2 - 2x|}$.

Find the points of discontinuities of f. What type of discontinuities are these?

Solution. The function $f(x) = \dfrac{(x+2) \cdot \sin^2(x-1)}{|x(x-1)(x+2)|}$ is continuous except the points $0, 1, -2$. **(1p)**

- $\lim\limits_{x \to 0} f(x) = \lim\limits_{x \to 0} \dfrac{(x+2)\sin^2(x-1)}{|x^2 + x - 2|} \cdot \dfrac{1}{|x|} = +\infty$ **(3p)**

  $\Longrightarrow$ f has an essential discontinuity at $x = 0$. **(2p)**

- $\lim\limits_{x \to 1} f(x) = \lim\limits_{x \to 1} \dfrac{(x+2)}{|x^2 + 2x|} \cdot \dfrac{\sin^2(x-1)}{(x-1)^2} \cdot |x - 1| = 0$ **(4p)**

  $\Longrightarrow$ f has a removable discontinuity at $x = 1$ **(2p)**

- $\lim\limits_{x \to -2\pm} f(x) = \lim\limits_{x \to 1} \dfrac{(x+2)}{|x+2|} \cdot \dfrac{\sin^2(x-1)}{|x^2 - x|} = \pm \dfrac{\sin^2 3}{6}$ **(4p)**

  $\Longrightarrow$ f has a jump discontinuity at $x = -2$ **(2p)**

3. (25 points) Let $f(x) = \sqrt[3]{x(x+2)} \cdot \sin\left(\dfrac{\pi}{4} x\right)$.

Investigate whether the following tangent lines of f exist. If so, give their equations.

a) The tangent line at $x_0 = 2$.

b) The tangent line at $x_0 = 0$.

In case b), use the definition to calculate $f'(0)$.

Solution. a) If $x \notin \{0, -2\}$, then

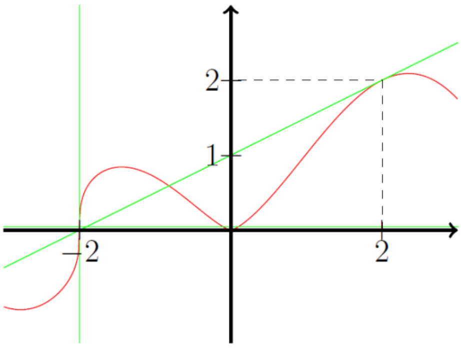
$$f'(x) = \frac{1}{3\sqrt[3]{(x(x+2))^2}}(2x + 2)\sin\left(\frac{\pi}{4}x\right) + \sqrt[3]{x(x+2)}\frac{\pi}{4}\cos\left(\frac{\pi}{4}x\right) \quad \textbf{(8p)}$$

so $f'(2) = \frac{1}{2}$ and $f(2) = 2$ and thus the tangent line is $y = f(2) + f'(2)(x-2) = \frac{x}{2} + 1$. **(7p)**

b)

$$f'(0) = \lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \underbrace{\sqrt[3]{x(x+2)}}_{\to 0} \underbrace{\frac{\sin\left(\frac{\pi}{4}x\right)}{\frac{\pi}{4}x}}_{\to 1} \frac{\pi}{4} = 0$$

and $f(0) = 0$, so the tangent line is $y = 0$ **(10p)**



4. (10+5+10 points) Calculate the following limits:

a) $\lim_{x\to 0} \frac{e^{-x}-1}{\arctan x}$ **b)** $\lim_{x\to\infty} \frac{e^{-x}-1}{\arctan x}$ **c)** $\lim_{x\to 1+0} \left(\ln(x-1)\cdot\cot\left(\frac{\pi}{2}x\right)\right)$

Solution.

$$\text{(a)} \quad \lim_{x\to 0} \frac{e^{-x}-1}{\operatorname{arctg} x} \overset{\frac{0}{0}\text{L'H}}{=\!=} \lim_{x\to 0} \frac{-e^{-x}}{\frac{1}{x^2+1}} = -1 \ \boxed{\mathbf{10p.}}$$

$$\text{(b)} \quad \lim_{x\to\infty} \frac{e^{-x}-1}{\operatorname{arctg} x} = -\frac{2}{\pi} \ \boxed{\mathbf{5p.}}$$

$$\text{(c)} \quad \lim_{x\to 1+} \left(\ln(x-1)\cdot\operatorname{ctg}\left(\frac{\pi}{2}x\right)\right) = \lim_{x\to 1+} \frac{\ln(x-1)}{\operatorname{tg}\left(\frac{\pi}{2}x\right)} \overset{\frac{-\infty}{-\infty}\text{L'H}}{=\!=} \lim_{x\to 1+} \frac{\frac{1}{x-1}}{\frac{1}{\cos^2\left(\frac{\pi}{2}x\right)}\cdot\frac{\pi}{2}} =$$

$$\lim_{x\to 1+} \frac{2\cos^2\left(\frac{\pi}{2}x\right)}{\pi(x-1)} \overset{\frac{0}{0}\text{L'H}}{=\!=} \lim_{x\to 1+} \frac{-4\cos\left(\frac{\pi}{2}x\right)\cdot\sin\left(\frac{\pi}{2}x\right)\frac{\pi}{2}}{\pi} = 0 \ \boxed{\mathbf{10p.}}$$

5. (18 points) Analyze the following function and plot its graph: $f(x) = \frac{(x+2)^2}{x^2}$.

Solution. • Domain: $D_f = \mathbb{R}\setminus\{0\}$. • Zeros: $f(x) = 0 \iff x = -2$

• Limits: $\lim_{x\to 0\pm 0} f(x) = \frac{4}{0+} = +\infty$, $\lim_{x\to\pm\infty} f(x) = 1$. **(1p)**

• $f'(x) = -\frac{4(x+2)}{x^3} = 0 \iff x = -2$ **(3p)**

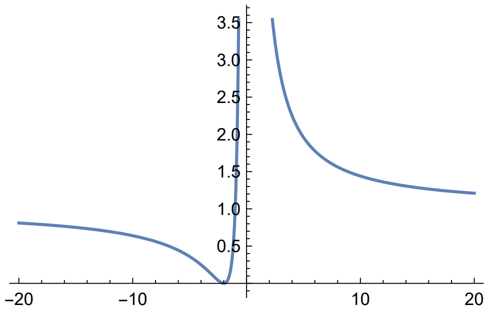
x	x<-2	x=-2	-2<x<0	x=0	x>0
f'	-	0	+	not defined	-
f	↘	loc. min. f(-2) = 0	↗	not defined	↘

(4p)

• $f''(x) = \dfrac{8(x+3)}{x^4} = 0 \iff x = -3$ **(3p)**

x	x<-3	x=-3	-3<x<0	x=0	0<x
f''	-	0	+	not defined	+
f	∩	infl: $f(-3) = \frac{1}{9}$	∪	not defined	∪

(4p)



(3p)

6. (12 points) We want to make a box with an open top from a 1 m by 1 m piece of cardboard, such that we cut out congruent squares from its corners, and turn up the sides. What are the sides of the box with maximal volume? What is this maximal volume?

Solution. Let x denote the sides of the squares. Then the base of the box is a square with sides $1-2x$ and the height of the box is x. The volume of the box is $V(x) = x(1-2x)^2 = 4x^3 - 4x^2 + x$.

We want to find the maximum of this function if $0 < x < \frac{1}{2}$. **(3p)**

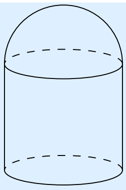
$V'(x) = 12x^2 - 8x + 1 = 0 \iff x_1 = \frac{1}{6}, x_2 = \frac{1}{2}$ **(3p)**

Because of the conditions, $x = \frac{1}{2}$ cannot be the case.

$V''(x) = 24x - 8$ and $V''\left(\frac{1}{6}\right) = 24 \cdot \frac{1}{6} - 8 = -4 < 0$, so V has a maximum at $x = \frac{1}{6}$. **(2p)**

The sides of the box with maximal volume are $\frac{2}{3}$, $\frac{2}{3}$ and $\frac{1}{6}$ m and the maximum of the volume is $\frac{2}{27}$ m^3. **(2p)**

7.* (10 points - BONUS)
We would like to pour oil into a bottle that is cylindrical at the bottom and hemispherical at the top, for which we have plastic for a total surface area of 30 dm^2. Determine the radius of the base circle of the cylinder so that as much oil as possible fits into the bottle.



Solution. Let r and h denote the radius and height of the cylinder, respectively. Then the volume and surface area of the bottle are

$$V = r^2 \pi h + \frac{2}{3} r^3 \pi, \qquad A = r^2 \pi + 2 r h \pi + 2 r^2 \pi \quad \textbf{(2p)}$$

Since $A = 30\ \text{dm}^2$ is given, then expressing h from the second equation and substituting into the first one we get

$$V(r) = r^2 \pi \cdot \frac{A - 3r^2\pi}{2r\pi} + \frac{2}{3} r^3 \pi = \frac{Ar - 3r^3\pi}{2} + \frac{2}{3} r^3 \pi \quad \textbf{(3p)}$$

It can be seen that the function $V(r)$ is differentiable on the interval $(0, \infty)$ (in reality, r is bounded above, since the bottom of the bottle cannot be larger than $10\ \text{dm}^2$), therefore at the points where V has a local extremum, its derivative is zero:

$$V'(r) = \frac{A}{2} - \frac{9r^2\pi}{2} + 2r^2\pi = \frac{A}{2} - \frac{5r^2\pi}{2} \quad \textbf{(2p)}$$

It has a unique positive root at $r_0 = \sqrt{\frac{A}{5\pi}} = \sqrt{\frac{6}{\pi}}\ \text{dm}$

Since the derivative is positive before r_0 and negative after it, then V has a local maximum at r_0. **(3p)**