Calculus 1, Repeated midterm test 1

11th December, 2023

Name: ______________________________ **Neptun code:** __________

1.	2.	3.	4.	5.	6.	7.	8.	9.	$\sum$

1. (10 points) Let $a_n = \dfrac{2n^3 - 3n}{n^3 + n + 2}$. Find the limit of a_n and provide a threshold index N for $\varepsilon = 0.01$.

2. (10 points) Find the limit of the following sequence: $c_n = \sqrt{n^3 - 5n - 3} - \sqrt{n^3 - 2n + 6}$

3. (10+10 points) Find the limit of the following sequences:

a) $a_n = \sqrt[n]{\dfrac{7^n + 5^n}{n^3 + 2}}$ b) $b_n = \left(\dfrac{2n^2 - 3}{2n^2 + 4}\right)^{3n^2}$

4. (4+4+4 points) Let $a_1 = 5$ and $a_{n+1} = \sqrt{10a_n - 21}$ for all $n \in \mathbb{N}$.

a) Prove that $3 < a_n < 7$ for all $n \in \mathbb{N}$.

b) Prove that the sequence is monotonically increasing.

c) Calculate the limit of the sequence (a_n).

5. (12 points) Find the liminf and limsup of $a_n = \left(\dfrac{n + (-1)^n}{n + 2}\right)^{n+3}$. Is the sequence convergent?

6. (6 points) Calculate the sum of the following series: $\displaystyle\sum_{n=1}^{\infty} \frac{2^{3n+1} + (-5)^{n-1}}{3^{2n+2}}$.

7. (10+10 points) Decide whether the following series are convergent or divergent:

a) $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)! \cdot 5^{n-1}}{(3n)!}$ b) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n+3}{2n+3}\right)^{n^2}$

8. (10 points) For what values of $x \in \mathbb{R}$ does the following series converge?

$\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{(n+3) \cdot 2^n} (x+5)^n$

9.* (10 points - BONUS): Let $a_0 = 1$ and $a_{n+1} = -\dfrac{1}{2} \cdot \sqrt[3]{a_n}$.

Find the limit points of this recursive sequence. (Help: Investigate the subsequences with odd and even indexes.)

Solutions

1. (10 points) Let $a_n = \dfrac{2n^3 - 3n}{n^3 + n + 2}$. Find the limit of a_n and provide a threshold index N for $\varepsilon = 0.01$.

Solution. $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \dfrac{n^3}{n^3} \cdot \dfrac{2 - \frac{3}{n^2}}{1 + \frac{1}{n^2} + \frac{2}{n^3}} = \dfrac{2-0}{1+0+0} = 2$ **(2p)**

We have to find $N(\varepsilon) \in \mathbb{N}$, such that if $n > N$, then $|a_n - A| < \varepsilon$ $(A = 2)$. **(1p)**

Let $\varepsilon > 0$. Then $\left|\dfrac{2n^3 - 3n}{n^3 + n + 2} - 2\right| = \dfrac{5n + 4}{n^3 + n + 2} \le \dfrac{9n}{n^3} = \dfrac{9}{n^2} < \varepsilon$ **(5p)**

if $n > \sqrt{\dfrac{9}{\varepsilon}}$, so with the choice $N(\varepsilon) = \left[\sqrt{\dfrac{9}{\varepsilon}}\right]$ the definition holds. **(1p)**

If $\varepsilon = 0.01$, then $N \ge \left[\sqrt{\dfrac{9}{0.01}}\right] = 30$. **(1p)**

2. (10 points) Find the limit of the following sequence: $c_n = \sqrt{n^3 - 5n - 3} - \sqrt{n^3 - 2n + 6}$

Solution.

$$c_n \overset{(3p)}{=} \frac{n^3 - 5n - 3 - (n^3 - 2n + 6)}{\sqrt{n^3 - 5n - 3} + \sqrt{n^3 - 2n + 6}} \overset{(2p)}{=} \frac{-3n - 9}{\sqrt{n^3 - 5n - 3} + \sqrt{n^3 - 2n + 6}} \overset{(3p)}{=}$$

$$= \frac{n}{n^{3/2}} \cdot \frac{-3 - \frac{9}{n}}{\sqrt{1 - \frac{5}{n^2} - \frac{3}{n^3}} + \sqrt{1 - \frac{2}{n^2} + \frac{6}{n^3}}} \overset{(2p)}{\to} 0.$$

3. (10+10 points) Find the limit of the following sequences:

a) $a_n = \sqrt[n]{\dfrac{7^n + 5^n}{n^3 + 2}}$ b) $b_n = \left(\dfrac{2n^2 - 3}{2n^2 + 4}\right)^{3n^2}$

Solution. a) Because of the Sandwich Theorem, $\lim\limits_{n\to\infty} \sqrt[n]{\dfrac{7^n + 5^n}{n^3 + 2}} = 7$ **(2p)**, since

$$7 \overset{(1p)}{\leftarrow} \frac{7}{\sqrt[n]{3}\,(\sqrt[n]{n})^3} \overset{(1p)}{=} \sqrt[n]{\frac{7^n}{3n^3}} \overset{(2p)}{\le} \sqrt[n]{\frac{7^n + 5^n}{n^3 + 2}} \overset{(2p)}{\le} \sqrt[n]{\frac{2 \cdot 7^n}{n^3}} \overset{(1p)}{=} \frac{7\sqrt[n]{2}}{(\sqrt[n]{n})^3} \overset{(1p)}{\to} 7$$

b) $$b_n = \left(\frac{\left(1 - \frac{3}{2n^2}\right)^{n^2}}{\left(1 + \frac{4}{2n^2}\right)^{n^2}}\right)^3 \to \left(\frac{e^{-\frac{3}{2}}}{e^{\frac{4}{2}}}\right)^3 = e^{-\frac{21}{2}} \quad \mathbf{(5+4+1p)}$$

4. (4+4+4 points) Let $a_1 = 5$ and $a_{n+1} = \sqrt{10\,a_n - 21}$ for all $n \in \mathbb{N}$.

a) Prove that $3 < a_n < 7$ for all $n \in \mathbb{N}$.

b) Prove that the sequence is monotonically increasing.

c) Calculate the limit of the sequence (a_n).

Solution. a) Boundedness: we prove by induction that $3 < a_n < 7$ for all $n \in \mathbb{N}$.

(1) $3 < a_1 = 5 < 7$

(2) Assume that $3 < a_n < 7$. We need to show that this implies $3 < a_{n+1} < 7$ $(n \in \mathbb{N})$.

(3) Then $30 - 21 < 10\,a_n - 21 < 70 - 21 \implies 9 < 10\,a_n - 21 < 49 \implies 3 < \sqrt{10\,a_n - 21} < 7$

So (a_n) is bounded above. **(4p)**

b) Monotonicity: we prove by induction that (a_n) is monotonically increasing, that is, $a_n < a_{n+1}$ $\forall\, n \in \mathbb{N}$.

(1) $a_1 = 5 < a_2 = \sqrt{50 - 21} = \sqrt{29}$

(2) Assume that $a_n < a_{n+1}$

(3) Then $10\,a_n - 21 < 10\,a_{n+1} - 21 \implies a_{n+1} = \sqrt{10\,a_n - 21} < \sqrt{10\,a_{n+1} - 21} = a_{n+2} \implies a_{n+1} < a_{n+2}$

So (a_n) is monotonically increasing. **(4p)**

c) Since (a_n) is monotonically increasing and bounded above then it is convergent.

Let $\lim\limits_{n\to\infty} a_n = A$. Then $A = \sqrt{10\,A - 21} \iff A^2 - 10\,A + 21 = (A-3)(A-7) = 0 \iff A_1 = 3, A_2 = 7$.

Since $a_1 = 5$ and the sequence is monotonically increasing then $A = 3$ cannot be the limit.

So $\lim\limits_{n\to\infty} a_n = 7$. **(4p)**

5. (12 points) Find the liminf and limsup of $a_n = \left(\dfrac{n + (-1)^n}{n+2}\right)^{n+3}$. Is the sequence convergent?

Solution. If n is even, then $a_n = \left(\dfrac{n+1}{n+2}\right)^{n+3} = \dfrac{\left(1+\frac{1}{n}\right)^n}{\left(1+\frac{2}{n}\right)^n} \cdot \left(\dfrac{n+1}{n+2}\right)^3 \to \dfrac{1}{e}$ **(5p)**

If n is odd, then $a_n = \left(\dfrac{n-1}{n+2}\right)^{n+3} = \dfrac{\left(1-\frac{1}{n}\right)^n}{\left(1+\frac{2}{n}\right)^n} \cdot \left(\dfrac{n-1}{n+2}\right)^3 \to \dfrac{1}{e^3}$ **(4p)**

Therefore, the limit points of the sequence are $\frac{1}{e}$ and $\frac{1}{e^3}$, so $\liminf a_n = \frac{1}{e^3}$, $\limsup a_n = \frac{1}{e}$ **(2p)**,

Since $\liminf a_n \neq \limsup a_n$, then the sequence is divergent. **(1p)**

6. (6 points) Calculate the sum of the following series: $\displaystyle\sum_{n=1}^{\infty} \frac{2^{3n+1} + (-5)^{n-1}}{3^{2n+2}}$.

Solution. $\displaystyle\sum_{n=1}^{\infty} \frac{2^{3n+1} + (-5)^{n-1}}{3^{2n+2}} = \sum_{n=2}^{\infty} \frac{2 \cdot 8^n + \left(-\frac{1}{5}\right)(-5)^n}{9 \cdot 9^n} = \sum_{n=2}^{\infty} \left(\frac{2}{9} \cdot \left(\frac{8}{9}\right)^n - \frac{1}{45} \cdot \left(\frac{-5}{9}\right)^n\right) =$ **(2p)**

$= \dfrac{2}{9}\,\dfrac{\frac{8}{9}}{1-\frac{8}{9}} - \dfrac{1}{45}\,\dfrac{-\frac{5}{9}}{1-\left(-\frac{5}{9}\right)}$ **(4p)** $= \dfrac{16}{9} + \dfrac{1}{126} = \dfrac{25}{14}$

7. (10+10 points) Decide whether the following series are convergent or divergent:

a) $\displaystyle\sum_{n=1}^{\infty} \frac{(2\,n)! \cdot 5^{n-1}}{(3\,n)!}$ b) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n+3}{2\,n+3}\right)^{n^2}$

Solution.

a) Let $a_n = \dfrac{(2\,n)! \cdot 5^{n-1}}{(3\,n)!}$. By the ratio test:

$$\frac{a_{n+1}}{a_n} = \frac{(2n+2)!\cdot 5^n}{(3n+3)!}\cdot\frac{(3n)!}{(2n)!\cdot 5^{n-1}}\quad \textbf{(3p)}$$

$$= \frac{(2n+1)(2n+2)\cdot 5}{(3n+1)(3n+2)(3n+3)} = \frac{n^2}{n^3}\cdot\frac{\left(2+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)\cdot 5}{\left(3+\frac{1}{n}\right)\left(3+\frac{2}{n}\right)\left(3+\frac{3}{n}\right)} \longrightarrow 0\cdot\frac{2\cdot 2\cdot 5}{3\cdot 3\cdot 3} = 0 < 1\quad \textbf{(5p)}$$

$\Longrightarrow$ the series $\sum_{n=1}^{\infty} a_n$ is convergent **(1p)**

b) Let $a_n = \left(\frac{n+3}{2n+3}\right)^{n^2}$. By the root test:

$$\sqrt[n]{a_n} = \left(\frac{n+3}{2n+3}\right)^n\ \textbf{(3p)}\ = \left(\frac{n}{2n}\cdot\frac{1+\frac{3}{n}}{1+\frac{3}{2n}}\right)^n = \left(\frac{1}{2}\right)^n\cdot\frac{\left(1+\frac{3}{n}\right)^n}{\left(1+\frac{3}{2n}\right)^n} \longrightarrow 0\cdot\frac{e^3}{e^{\frac{3}{2}}} = 0 < 1\quad \textbf{(5p)}$$

$\Longrightarrow$ the series $\sum_{n=1}^{\infty} a_n$ is convergent **(1p)**

8. (10 points) For what values of $x \in \mathbb{R}$ does the following series converge?
$\sum_{n=1}^{\infty} \frac{(-1)^n}{(n+3)\cdot 2^n}(x+5)^n$

Solution. The coefficients are $a_n = \frac{(-1)^n}{(n+3)\cdot 2^n}$ and the center is $x_0 = -5$.

$$\sqrt[n]{|\,a_n\,|} = \sqrt[n]{\frac{1}{(n+3)\cdot 2^n}} = \frac{1}{\sqrt[n]{n+3}\cdot 2} \longrightarrow \frac{1}{1\cdot 2} = \frac{1}{2} = \frac{1}{R} \Longrightarrow R = 2$$

Here we used that $\sqrt[n]{n+3} \longrightarrow 1$ by the Sandwich Theorem, since
$1 \le \sqrt[n]{n+3} \le \sqrt[n]{n+3n} = \sqrt[n]{4}\cdot\sqrt[n]{n} \longrightarrow 1\cdot 1 = 1.$ **(5p)**

Let H denote the domain of convergence. The endpoints of H:

If $x = x_0 - R = -5-2 = -7$ then the series is $\sum_{n=1}^{\infty}\frac{(-1)^n}{(n+3)\cdot 2^n}\cdot(-2)^n = \sum_{n=1}^{\infty}\frac{1}{n+3}$.

Since $\frac{1}{n+3} \ge \frac{1}{n+3n} = \frac{1}{4n}$ and $\sum_{n=1}^{\infty}\frac{1}{4}\cdot\frac{1}{n}$ diverges, then by the comparison test,
$\sum_{n=1}^{\infty}\frac{1}{n+3}$ also diverges. $\Longrightarrow -7 \notin H$. **(2p)**

If $x = x_0 + R = -5+2 = -3$ then the series is $\sum_{n=1}^{\infty}\frac{(-1)^n}{(n+3)\cdot 2^n}\cdot 2^n = \sum_{n=1}^{\infty}\frac{(-1)^n}{(n+3)}$

This is a Leibniz series, so it is convergent $\Longrightarrow$ $-3 \in H$. **(2p)**
The domain of convergence is $H = (-7, -3]$. **(1p)**

9.* (10 points - BONUS): Let $a_0 = 1$ and $a_{n+1} = -\frac{1}{2}\cdot\sqrt[3]{a_n}$.

Find the limit points of this recursive sequence. (Help: Investigate the subsequences with odd and even indexes.)

Solution. Using that n and $n+2$ have the same parity, for the subsequences with odd and even indexes we have

$a_{n+2} = -\frac{1}{2} \cdot \sqrt[3]{a_{n+1}} = -\frac{1}{2} \cdot \sqrt[3]{-\frac{1}{2} \cdot \sqrt[3]{a_n}} = \frac{1}{2 \cdot \sqrt[3]{2}} \cdot \sqrt[9]{a_n} = 2^{-\frac{4}{3}} \cdot \sqrt[9]{a_n}$ **(2p)**

Solutions of $A = 2^{-\frac{4}{3}} \cdot \sqrt[9]{A}$ are $A_0 = 0$, $A_\pm = \pm 2^{-\frac{3}{2}} = \pm \frac{1}{2\sqrt{2}}$ **(3p)**

Observing that a_{2n} is monotonically decreasing and tends to A_+: **(2p)**
Observing that a_{2n+1} is monotonically increasing and tends to A_-: **(2p)**
Therefore the limit points are A_+ and A_-. **(1p)**