Calculus 1, Final exam 2, Part 2

12th January, 2024

Name:______________________________ Neptun code:__________

1.:_____ 2.:_____ 3.:_____ 4.:_____ 5.:_____ 6.:_____ 7.:_____ 8.:_____ Sum:_____

1. (9+6 points) Calculate the following limits:

a) $\lim_{x\to 0} \frac{2x(e^{3x}-1)}{\sin(6x^2)}$ b) $\lim_{x\to 0} (1+\sin(3x))^{\frac{1}{x}}$

2. (5+5 points) Calculate the derivatives of the following functions:

a) $f(x) = \sinh\left(\sqrt{\frac{1+e^{\sin^2 x+1}}{2+\sin(4x)}}\right)$ b) $g(x) = \left(\frac{2x-1}{x^2+3}\right)\tan\left(\sqrt{\frac{e^{x^2}+2}{x^2+1}}\right)$

3. (10 points) Estimate the value of $\sqrt[3]{1.5}$ using the second-order Taylor-polynomial of the function $f:\mathbb{R}\longrightarrow\mathbb{R}$, $f(x) = \sqrt[3]{1+x}$ around the point $x_0 = 0$, and prove that the error of this estimation is less than 0.01.

4. (15 points) Analyze the following function and sketch its graph: $f(x) = e^{-x}(x^2+2x-1)$.

5. (10+10 points) Calculate the following integrals:

a) $I_1 = \int_0^{\ln 2} (2x-1)\, e^{-2x}\, \mathrm{dx}$ b) $I_2 = \int \arctan\left(\sqrt{x}\right) \mathrm{dx}$ (substitution: $t = \sqrt{x}$)

6. (10+10 points) Calculate the following integrals:

a) $I_3 = \int \frac{x+3}{x(x+1)^2}\, \mathrm{dx}$ b) $I_4 = \int \frac{1}{e^{2x}+1}\, \mathrm{dx}$ (substitution: $t = e^x$)

7. (10 points) Consider the function $f(x) = \frac{\sqrt{\sin x}}{\cos x + 2}$ on the interval $x \in [0, \pi]$.
Rotate it around the x-axis and find the volume of the arising body.

8.* (10 points - BONUS) What can the area of a right-angled triangle be at most, if the sum of its one leg and its hypotenuse is 10 cm?

Solutions

1. (9+6 points) Calculate the following limits:

a) $\lim\limits_{x\to 0} \dfrac{2x(e^{3x}-1)}{\sin(6x^2)}$ b) $\lim\limits_{x\to 0} (1+\sin(3x))^{\frac{1}{x}}$

Solution. a) The limit has the form $\frac{0}{0}$, so the L'Hospital's rule can be applied:

$$\lim_{x\to 0} \frac{2x(e^{3x}-1)}{\sin(6x^2)} \overset{\frac{0}{0},L'H}{=} \lim_{x\to 0} \frac{2(e^{3x}-1)+2x\cdot e^{3x}\cdot 3}{\cos(6x^2)\cdot 12x} \quad \textbf{(3p)} \quad \overset{\frac{0}{0},L'H}{=} \lim_{x\to 0} \frac{6e^{3x}+(6e^{3x}+6x\cdot e^{3x}\cdot 3)}{-\sin(6x^2)\cdot(12x)^2+\cos(6x^2)\cdot 12} \quad \textbf{(4p)}$$

$$= \frac{6+(6+0)}{0+1\cdot 12} = 1 \quad \textbf{(2p)}$$

b) $L = \lim\limits_{x\to 0} (1+\sin(3x))^{\frac{1}{x}} = \lim\limits_{x\to 0} e^{\ln\left((1+\sin(3x))^{\frac{1}{x}}\right)} = \lim\limits_{x\to 0} e^{\frac{\ln(1+\sin(3x))}{x}}$ **(2p)**

The limit has the form $\frac{0}{0}$, so the L'Hospital's rule can be applied:

$$\lim_{x\to 0} \frac{\ln(1+\sin(3x))}{x} \overset{\frac{0}{0},L'H}{=} \lim_{x\to 0} \frac{\frac{1}{1+\sin(3x)}\cdot\cos(3x)\cdot 3}{1} = \frac{\frac{1}{1+0}\cdot 1\cdot 3}{1} = 3 \quad \textbf{(3p)} \implies L = e^3 \quad \textbf{(1p)}$$

2. (5+5 points) Calculate the derivatives of the following functions:

a) $f(x) = \sinh\left(\sqrt{\dfrac{1+e^{\sin^2 x+1}}{2+\sin(4x)}}\right)$ b) $g(x) = \left(\dfrac{2x-1}{x^2+3}\right)\tan\left(\sqrt{\dfrac{e^{x^2}+2}{x^2+1}}\right)$

Solution.

a) $$f'(x) = \cosh\left(\sqrt{\frac{1+e^{\sin^2 x+1}}{2+\sin(4x)}}\right)\cdot\frac{1}{2}\left(\frac{1+e^{\sin^2 x+1}}{2+\sin(4x)}\right)^{-\frac{1}{2}}\cdot$$
$$\cdot\frac{\left(e^{\sin^2 x+1}\cdot 2\sin x\cos x\right)(2+\sin(4x))-\left(1+e^{\sin^2 x+1}\right)(\cos 4x\cdot 4)}{(2+\sin(4x))^2}$$

b) $$g'(x) = \frac{2(x^2+3)-(2x-1)\cdot 2x}{(x^2+3)^2}\cdot\tan\left(\sqrt{\frac{e^{x^2}+2}{x^2+1}}\right)+$$
$$+\left(\frac{2x-1}{x^2+3}\right)\cdot\frac{1}{\cos^2\left(\sqrt{\frac{e^{x^2}+2}{x^2+1}}\right)}\cdot\frac{1}{2}\left(\frac{e^{x^2}+2}{x^2+1}\right)^{-\frac{1}{2}}\cdot\frac{\left(e^{x^2}\cdot 2x\right)(x^2+1)-\left(e^{x^2}+2\right)(2x)}{(x^2+1)^2}$$

3. (10 points) Estimate the value of $\sqrt[3]{1.5}$ using the second-order Taylor-polynomial of the function $f:\mathbb{R}\longrightarrow\mathbb{R}$, $f(x)=\sqrt[3]{1+x}$ around the point $x_0=0$, and prove that the error of this estimation is less than 0.01.

Solution.
$$f(x)=\sqrt[3]{1+x} \quad\Longrightarrow\quad f(0)=1$$
$$f'(x)=\frac{1}{3}(1+x)^{-\frac{2}{3}} \quad\Longrightarrow\quad f'(0)=\frac{1}{3}$$
$$f''(x)=-\frac{2}{9}(1+x)^{-\frac{5}{3}} \quad\Longrightarrow\quad f''(0)=-\frac{2}{9}$$
$$f'''(x)=\frac{10}{27}(1+x)^{-\frac{8}{3}} \qquad \textbf{(3p)}$$

The second order Taylor polynomial of f around $x_0=0$ is
$$T_2(x)=f(0)+f'(0)\,x+\frac{f''(0)}{2!}x^2=1+\frac{1}{3}x+\frac{-\frac{2}{9}}{2!}x^2=1+\frac{1}{3}x-\frac{1}{9}x^2$$
If $x=0.5$ then substituting into the Taylor polynomial we get an estimation for $\sqrt[3]{1.5}$:
$$f(0.5)=\sqrt[3]{1.5}\approx T_2(0.5)=1+\frac{1}{3}\cdot 0.5-\frac{1}{9}\cdot 0.5^2 \qquad \textbf{(3p)}$$

Lagrange remainder term: $R_n(x)=\dfrac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}$, where $n=2$, $x_0=0$, $x=0.5$, $0<\xi<0.5$

Taylor's theorem: $f(x)=T_n(x)+R_n(x)$

The error for the approximation $f(x)\approx T_2(x)$ can be estimate from above:
$$|E| = |f(x)-T_2(x)| = |R_2(x)| = \left|\frac{f^{(3)}(\xi)}{3!}(0.5-0)^3\right| = \left|\frac{1}{3!}\cdot\frac{10}{27}\cdot\frac{1}{(1+\xi)^{\frac{8}{3}}}\cdot 0.5^3\right|$$
$$=\frac{1}{3!}\cdot\frac{10}{27}\cdot\frac{1}{(1+\xi)^{\frac{8}{3}}}\cdot\left(\frac{1}{2}\right)^3 \le \frac{1}{3!}\cdot\frac{10}{27}\cdot\frac{1}{1}\cdot\left(\frac{1}{2}\right)^3=\frac{10}{6\cdot 27\cdot 8}\ \textbf{(4p)}\ =\frac{5}{648}\approx 0.00771605$$

Remark. Comparison of the numerical values:

$f(0.5)\approx 1.14471$

$T_2(0.5)\approx 1.13889$

$f(0.5)-T_2(0.5)\approx 0.00582535$

4. (15 points) Analyze the following function and sketch its graph: $f(x)=e^{-x}\left(x^2+2x-1\right)$.

Solution.

$D_f=\mathbb{R}$; $\left(f(x)=0 \iff x=-1\pm\sqrt{2}\right)$
$$\lim_{x\to+\infty} f(x)=\lim_{x\to+\infty}\frac{x^2+2x-1}{e^x}\overset{L'H}{=}\lim_{x\to+\infty}\frac{2x+2}{e^x}\overset{L'H}{=}\lim_{x\to+\infty}\frac{2}{e^x}=0$$
$$\lim_{x\to-\infty} f(x)=\infty\cdot\infty=\infty \quad \textbf{(2p)}$$

(1) Monotonicity, local extrema
$$f'(x)=e^{-x}\left(-x^2+3\right)=0 \iff x=\pm\sqrt{3}\approx\pm 1.73205 \quad \textbf{(2p)}$$

x	$x<-\sqrt{3}$	$x=-\sqrt{3}$	$-\sqrt{3}<x<\sqrt{3}$	$x=\sqrt{3}$	$x>\sqrt{3}$
f'	–	0	+	0	–
f	↘	loc. min.	↗	loc. max.	↘

(3p)

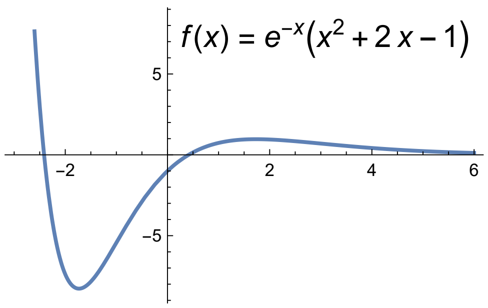
(2) Convexity, inflection points

$f''(x) = e^{-x}(x-3)(x+1) = 0 \iff x = -1$ or $x = 3$ **(2p)**

x	x<-3	x=-3	-3<x<1	x=1	x>1
f''	+	0	–	0	+
f	∪	infl.	∩	infl.	∪

(3p)

(3) The graph of f, taking into account the limits at $\pm\infty$ **(3p)**



5. (10+10 points) Calculate the following integrals:

a) $I_1 = \int_0^{\ln 2} (2x-1)\, e^{-2x}\, dx$ b) $I_2 = \int \arctan(\sqrt{x})\, dx$ (substitution: $t = \sqrt{x}$)

Solution. a) $I_1 = \int_0^{\ln 2} (2x-1)\, e^{-2x}\, dx = ?$

First we calculate the indefinite integral with integration by parts:

$$\int (2x-1)\, e^{-2x}\, dx = \frac{e^{-2x}}{-2} \cdot (2x-1) - \int \frac{e^{-2x}}{-2} \cdot 2\, dx \quad \textbf{(3p)}$$

$$= \frac{e^{-2x}}{-2} \cdot (2x-1) + \int e^{-2x}\, dx = \frac{e^{-2x}}{-2} \cdot (2x-1) + \frac{e^{-2x}}{-2} + c = \frac{e^{-2x}}{-2} (2x-1+1) + c = -x\, e^{-2x} + c \quad \textbf{(4p)}$$

The definite integral is $I_1 = \int_0^{\ln 2} (2x-1)\, e^{-2x}\, dx = \left[-x\, e^{-2x}\right]_0^{\ln 2} = \left(-\ln 2 \cdot e^{-2\ln 2} - 0\right) = -\frac{\ln 2}{4}$ **(3p)**

b) $I_2 = \int \arctan(\sqrt{x})\, dx = ?$ Substitution: $t = \sqrt{x} \implies x = x(t) = t^2 \implies x'(t) = \frac{dx}{dt} = 2t \implies dx = 2t\, dt$

$\implies I_2 = \int \arctan(t) \cdot 2t\, dt =$ **(3p)**

With integration by parts: $f'(t) = 2t \implies f(t) = t^2$

$g(t) = \arctan(t) \implies g'(t) = \frac{1}{t^2+1}$

$$I_2 = t^2 \arctan(t) - \int t^2 \cdot \frac{1}{t^2+1}\, dt \quad \textbf{(3p)} \quad = t^2 \arctan(t) - \int \frac{(t^2+1)-1}{t^2+1}\, dt =$$

$$= t^2 \arctan(t) - \int \left(1 - \frac{1}{t^2+1}\right) dt = t^2 \arctan(t) - (t - \arctan(t)) + c \quad \textbf{(3p)}$$

$= x \arctan\left(\sqrt{x}\right) - \sqrt{x} + \arctan\left(\sqrt{x}\right) + c$ **(1p)** $= (x+1)\arctan\left(\sqrt{x}\right) - \sqrt{x} + c$

6. (10+10 points) Calculate the following integrals:

a) $I_3 = \int \frac{x+3}{x(x+1)^2}\, dx$ b) $I_4 = \int \frac{1}{e^{2x}+1}\, dx$ (substitution: $t = e^x$)

Solution. a) We use partial fraction decomposition:

$\frac{x+3}{x(x+1)^2} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$ **(2p)** Multiplying by $x(x+1)^2$ we get:

$$x + 3 = A(x+1)^2 + B\,x(x+1) + C\,x$$

$x = 0 \implies 3 = A + 0 + 0 \implies A = 3$

$x = -1 \implies 2 = 0 + 0 - C \implies C = -2$ **(3p)**

$x = 1 \implies 4 = 4A + 2B + C \implies 2B = 4 - 4A - C = -6 \implies B = -3$

$\implies I_3 = \int \frac{x+3}{x(x+1)^2}\, dx = \int \left(\frac{3}{x} - \frac{3}{x+1} - \frac{2}{(x+1)^2}\right) dx =$

$= 3\ln|x| - 3\ln|x+1| - 2\cdot\frac{(x+1)^{-1}}{-1} + c$ **(5p)** $= 3\ln|x| - 3\ln|x+1| + \frac{2}{x+1} + c$

b) $I_4 = \int \frac{1}{e^{2x}+1}\, dx = ?$ (substitution: $t = e^x$)

Substitution: $t = e^x \implies x = x(t) = \ln t \implies x'(t) = \frac{dx}{dt} = \frac{1}{t} \implies dx = \frac{1}{t}\, dt$

$\implies I_4 = \int \frac{1}{t^2+1}\cdot\frac{1}{t}\, dt = \int \frac{1}{t\left(t^2+1\right)}\, dt$ **(3p)**

Partial fraction decomposition: $\frac{1}{t\left(t^2+1\right)} = \frac{A}{t} + \frac{B\,t + C}{t^2+1}$

$\implies 1 = A\left(t^2+1\right) + (B\,t + C)\,t$

$t = 0 \implies 1 = A + 0 \implies A = 1$

$t = 1 \implies 1 = 2A + B + C$

$t = -1 \implies 1 = 2A + B - C \implies C = 0,\ B = -1$ **(4p)**

$\implies I_4 = \int \left(\frac{1}{t} - \frac{t}{t^2+1}\right) dt = \int \left(\frac{1}{t} - \frac{1}{2}\frac{2t}{t^2+1}\right) dt = \ln|t| - \frac{1}{2}\ln\left(t^2+1\right) + c =$

$= \ln\left(e^x\right) - \frac{1}{2}\ln\left(e^{2x}+1\right) + c = x - \frac{1}{2}\ln\left(e^{2x}+1\right) + c$ **(3p)**

7. (10 points) Consider the function $f(x) = \frac{\sqrt{\sin x}}{\cos x + 2}$ on the interval $x \in [0, \pi]$.

Rotate it around the x-axis and find the volume of the arising body.

Solution. The volume is $V = \pi \int_0^\pi f^2(x)\, dx = \pi \int_0^\pi \frac{\sin x}{(\cos x + 2)^2}\, dx$ **(3p)**

$= \pi \int_0^{\pi} \sin x \cdot (\cos x + 2)^{-2} \, dx = \pi \int_0^{\pi} -(-\sin x) \cdot (\cos x + 2)^{-2} \, dx = \pi \left[-\frac{(\cos x + 2)^{-1}}{-1} \right]_0^{\pi}$ **(4p)** $= \pi \left[\frac{1}{\cos x + 2} \right]_0^{\pi}$

$= \pi \left[\frac{1}{-1+2} - \frac{1}{1+2} \right]$ **(2p)** $= \frac{2\pi}{3}$ **(1p)**

8.* (10 points - BONUS) What can the area of a right-angled triangle be at most, if the sum of its one leg and its hypotenuse is 10 cm?

Solution.

Let's denote the legs by x and y, and the hypotenuse by z. The hypotenuse is $z = 10 - x$, the other leg is

$$y = \sqrt{(10-x)^2 - x^2} = \sqrt{100 - 20x} = 2\sqrt{25 - 5x},$$

and the area of the triangle

$$T(x) = \frac{1}{2} xy = x\sqrt{25 - 5x}.$$

Since $0 \le x \le 5$, so we have to find the global maximum of the continuous function $T(x)$ on the closed interval $[0, 5]$. According to the Weierstrass theorem there is a maximum. Since $T(x) \ge 0$, $T(0) = T(5) = 0$, so the maximum is on the open interval $(0, 5)$. In this case the global maximum is also a local maximum, therefore according to theorem 4.9 the derivative is zero at this point. Let's find the roots of $T'(x)$:

$$T'(x) = \sqrt{25 - 5x} - \frac{5x}{2\sqrt{25 - 5x}} = 0$$

$$\sqrt{25 - 5x} = \frac{5x}{2\sqrt{25 - 5x}} \iff 2(25 - 5x) = 5x \iff x = \frac{10}{3}$$

Since the derivative has exactly one root, the (local) maximum can be only here. Therefore,

$$\max T = T\left(\frac{10}{3}\right) = \frac{10}{3}\sqrt{25 - \frac{50}{3}} = \frac{50}{3\sqrt{3}}.$$

The area is maximal, if

$$x = \frac{10}{3}, \quad y = \frac{10}{\sqrt{3}} = x\sqrt{3}, \quad z = \frac{20}{3} = 2x.$$

According to the given condition of the problem, the triangle which has maximal area, is the half of an equilateral triangle.

Remark: The function $T(x)$ has the maximum at the same point as the function $f(x) = T^2(x) = x^2(25-5x)$. But the maximum of $f(x)$ can be found by "smartly" applying the inequality between the arithmetic and geometric means. There is a 3-factor product on $(0, 5)$:

$$\left(\frac{5}{2}x\right)^2 (25 - 5x)$$

Here all of the factors are positive, and their sum is 25, independently of x. Therefore the product is maximal, if the factors are equal, that is,

$$\frac{5}{2}x = 25 - 5x, \quad \Longleftrightarrow \quad 15x = 50, \quad \Longleftrightarrow \quad x = \frac{10}{3}$$