

Calculus 1 - 15

Taylor polynomial, Taylor series

Taylor polynomial

Definition. Let f be at least n times differentiable at x_0 . Then the polynomial

$$T_n(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \dots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n = \sum_{k=0}^n \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

is the **n th Taylor polynomial of f at x_0** . (If $x_0 = 0$: Maclaurin polynomial.)

Example. Let $f(x) = \sin x$ and $x_0 = 0$.

$$f(x) = \sin x \quad \Rightarrow f(0) = 0 \quad \Rightarrow T_0(x) = 0$$

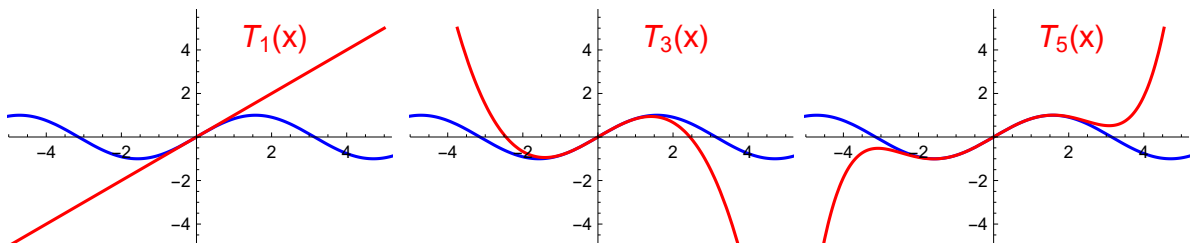
$$f'(x) = \cos x \quad f'(0) = 1 \quad T_1(x) = 0 + 1 \cdot x = x$$

$$f''(x) = -\sin x \quad f''(0) = 0 \quad T_2(x) = 0 + 1 \cdot x + \frac{0}{2!}x^2 = x$$

$$f'''(x) = -\cos x \quad f'''(0) = -1 \quad T_3(x) = 0 + 1 \cdot x + \frac{0}{2!}x^2 + \frac{-1}{3!}x^3 = x - \frac{x^3}{6}$$

$$f^{(4)}(x) = \sin x \quad f^{(4)}(0) = 0 \quad T_4(x) = 0 + 1 \cdot x + \frac{0}{2!}x^2 + \frac{-1}{3!}x^3 + \frac{0}{4!}x^4 = x - \frac{x^3}{6}$$

$$f^{(5)}(x) = \cos x \quad f^{(5)}(0) = 1 \quad T_5(x) = 0 + 1 \cdot x + \frac{0}{2!}x^2 + \frac{-1}{3!}x^3 + \frac{0}{4!}x^4 + \frac{1}{5!}x^5 = x - \frac{x^3}{6} + \frac{x^5}{120}$$



Theorem (Uniqueness of the Taylor polynomial).

Assume that f is at least n times differentiable at x_0 . Then $T_n(x)$ is the unique polynomial of degree at most n for which $T_n^{(k)}(x_0) = f^{(k)}(x_0)$ for $k = 0, 1, \dots, n$.

Proof. Express $T_n(x)$ as powers of $(x - x_0)$:

$$T_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + a_3(x - x_0)^3 + a_4(x - x_0)^4 + \dots + a_n(x - x_0)^n$$

$$\Rightarrow T_n(x_0) = a_0 = f(x_0)$$

$$T_n'(x) = a_1 + 2a_2(x - x_0) + 3a_3(x - x_0)^2 + 4a_4(x - x_0)^3 + \dots + na_n(x - x_0)^{n-1}$$

$$\Rightarrow T_n'(x_0) = a_1 = f'(x_0)$$

$$T_n''(x) = 2a_2 + 3 \cdot 2 \cdot a_3(x - x_0) + 4 \cdot 3 \cdot a_4(x - x_0)^2 + \dots + n(n-1)a_n(x - x_0)^{n-2}$$

$$\Rightarrow T_n''(x_0) = 2a_2 = f''(x_0)$$

$$T_n'''(x) = 3! \cdot a_3 + \dots + n(n-1)(n-2)a_n(x - x_0)^{n-3}$$

$$\Rightarrow T_n'''(x_0) = 3! a_3 = f'''(x_0)$$

Repeating the differentiation, we get that

$$T_n^{(k)}(x_0) = k! a_k = f^{(k)}(x_0), \dots, T_n^{(n)}(x_0) = n! a_n = f^{(n)}(x_0) \implies a_k = \frac{f^{(k)}(x_0)}{k!}, \quad k = 0, 1, \dots, n.$$

Theorem. If f is n times differentiable at x_0 then $\lim_{x \rightarrow x_0} \frac{f(x) - T_n(x)}{(x - x_0)^n} = 0$.

Proof. $\lim_{x \rightarrow x_0} \frac{f(x) - T_n(x)}{(x - x_0)^n} \stackrel{L'H}{=} \lim_{x \rightarrow x_0} \frac{f'(x) - T_n'(x)}{n(x - x_0)^{n-1}} \stackrel{L'H}{=} \lim_{x \rightarrow x_0} \frac{f''(x) - T_n''(x)}{n(n-1)(x - x_0)^{n-2}} \stackrel{L'H}{=} \dots \stackrel{L'H}{=} \lim_{x \rightarrow x_0} \frac{f^{(n)}(x) - T_n^{(n)}(x)}{n!(x - x_0)^{n-n}} = \frac{0}{n!} = 0$

Except the last one, the fractions are of the form $\frac{0}{0}$ so the L'Hospital's rule can be applied.

Taylor's theorem

Theorem (Taylor's theorem). Assume that f is at least $(n + 1)$ times differentiable on the interval $(x_0 - \delta, x_0 + \delta)$ and $x \in (x_0 - \delta, x_0 + \delta)$. Then there exists a number ξ between x and x_0 (that is, $x_0 < \xi < x$ or $x < \xi < x_0$) such that

$$R_n(x) = f(x) - T_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} (x - x_0)^{n+1}.$$

This expression is called the **Lagrange form of the remainder term**.

Proof. Assume that $x_0 < x$ and $y \in [x_0, x]$ (x is fixed).

Define the following function:

$$F(y) = f(y) + f'(y)(x - y) + \frac{f''(y)}{2!} (x - y)^2 + \dots + \frac{f^{(n)}(y)}{n!} (x - y)^n$$

Differentiating F with respect to y , we get a telescoping sum:

$$F'(y) = f'(y) + (f''(y)(x - y) - f'(y)) + \left(\frac{f'''(y)}{2!} (x - y)^2 - \frac{f''(y)}{2!} \cdot 2 \cdot (x - y) \right) + \dots + \left(\frac{f^{(n+1)}(y)}{n!} (x - y)^n - \frac{f^{(n)}(y)}{n!} \cdot n \cdot (x - y)^{n-1} \right) = \frac{f^{(n+1)}(y)}{n!} (x - y)^n$$

$$\implies F(x) = f(x)$$

$$F(x_0) = T_n(x_0)$$

$$F'(y) = \frac{f^{(n+1)}(y)}{n!} (x - y)^n$$

$$\text{Let } G(y) = (x - y)^{n+1} \implies G(x) = 0$$

$$G(x_0) = (x - x_0)^{n+1}$$

$$G'(y) = -(n+1)(x - y)^n$$

Both F and G are continuous on $[x_0, x]$, differentiable on (x_0, x) and $G'(y) \neq 0 \forall y \in (x_0, x)$, so by Cauchy's intermediate value theorem there exists $\xi \in (x_0, x)$ such that

$$\frac{f(x) - T_n(x_0)}{0 - (x - x_0)^{n+1}} = \frac{F(x) - F(x_0)}{G(x) - G(x_0)} = \frac{F'(\xi)}{G'(\xi)} = \frac{\frac{f^{(n+1)}(\xi)}{n!} (x - \xi)^n}{-(n+1)(x - \xi)^n} = -\frac{f^{(n+1)}(\xi)}{(n+1)!}$$

$$\implies f(x) - T_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} (x - x_0)^{n+1}.$$

Remark. If $n = 0$ then we obtain Lagrange's mean value theorem:

$$f(x) = T_0(x) + R_0(x) = f(x_0) + f'(\xi)(x - x_0)$$

Taylor series

Definition. If f is infinitely many times differentiable at x_0 then the power series

$$T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n \quad \text{is called the **Taylor series** of } f \text{ at } x_0.$$

Definition. The functions $g_0, g_1, \dots, g_n, \dots$ are **uniformly bounded** on the set $D = \bigcap D_{g_n}$ if

$$\exists K \in \mathbb{R}: |g_n(x)| \leq K \quad \text{if } x \in D, n \in \mathbb{N},$$

that is, the functions have a common bound K on the set D .

Example. The functions $\sin(x), 2\sin(2x), 3\sin(3x), \dots, n\sin(nx), \dots$ are each bounded but not uniformly bounded altogether.

The functions $\sin(x), \sin(2x), \sin(3x), \dots, \sin(nx), \dots$ are uniformly bounded, $K = 1$ is a suitable common bound.

Theorem. If f is infinitely many times differentiable on $(x_0 - R, x_0 + R)$ (R is the radius of convergence of T) and the functions $f, f', f'', f''', \dots, f^{(n)}, \dots$ are uniformly bounded on this interval then $f(x) = T(x)$ for all $x \in (x_0 - R, x_0 + R)$.

Proof. We have seen that $f(x) = T_n(x) + R_n(x) = \sum_{k=0}^n \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k + \frac{f^{(n+1)}(\xi)}{(n+1)!} (x - x_0)^{n+1}$.

Using uniform boundedness and the limit $\lim_{n \rightarrow \infty} \frac{a^n}{n!} = 0$, we can give an upper estimation

for the remainder term:

$$|f(x) - T_n(x)| = |R_n(x)| = \left| \frac{f^{(n+1)}(\xi)}{(n+1)!} (x - x_0)^{n+1} \right| = \frac{|f^{(n+1)}(\xi)|}{(n+1)!} |x - x_0|^{n+1} \leq K \cdot \frac{R^{n+1}}{(n+1)!} \xrightarrow{n \rightarrow \infty} 0$$

Therefore $T_n(x) \rightarrow f(x)$ and thus $f(x) = T(x)$.

Remark. Using this theorem, it can be shown that many elementary functions are equal to their power series.

Theorem. (1) $e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots, x \in \mathbb{R}$

(2) $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} + \dots, x \in \mathbb{R}$

(3) $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} + \dots, x \in \mathbb{R}$

(4) $\sinh x = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} x^{2n+1} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \frac{x^9}{9!} + \dots, x \in \mathbb{R}$

(5) $\cosh x = \sum_{n=0}^{\infty} \frac{1}{(2n)!} x^{2n} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \frac{x^8}{8!} + \dots, x \in \mathbb{R}$

In each case, the interval of convergence is the set of real numbers.

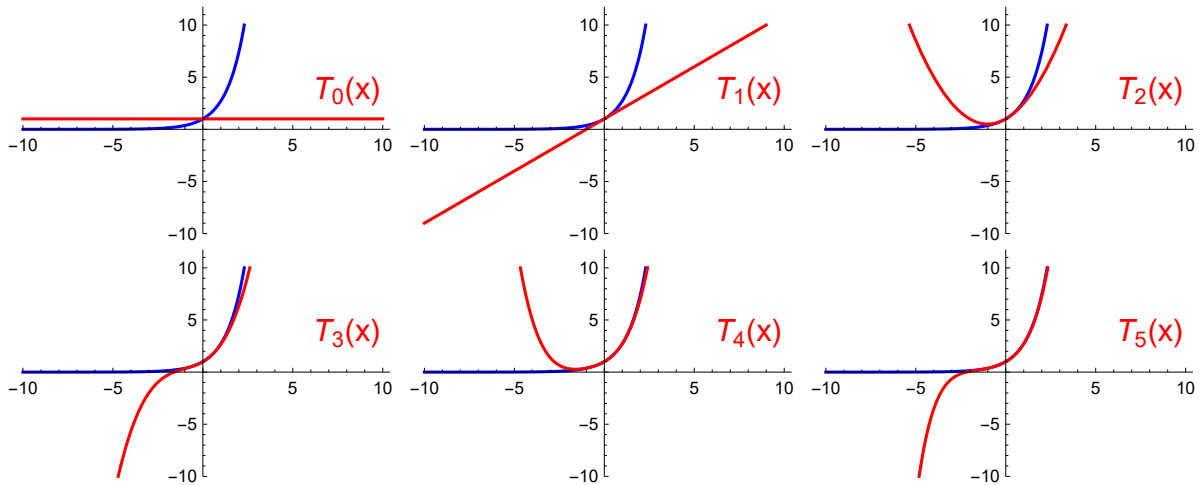
Proof. (1) Let $f(x) = e^x$ and $x_0 = 0 \implies f^{(k)}(x) = e^x \forall k \in \mathbb{N} \implies f^{(k)}(0) = 1$.

By the ratio test the radius of convergence is $\lim_{n \rightarrow \infty} \frac{\frac{1}{n!}}{\frac{1}{(n+1)!}} = \lim_{n \rightarrow \infty} \frac{(n+1)!}{n!} = \lim_{n \rightarrow \infty} (n+1) = \infty$

\implies the interval of convergence is \mathbb{R} .

The derivatives are not bounded on \mathbb{R} , however, they are uniformly bounded on any closed interval $[a, b]$ and $|f^{(k)}(x)| \leq e^b = K \implies f(x) = T(x)$ if $x \in [a, b]$.

Since this equality holds for **any** closed interval $[a, b]$ then $f(x) = T(x) \forall x \in \mathbb{R}$ also holds.

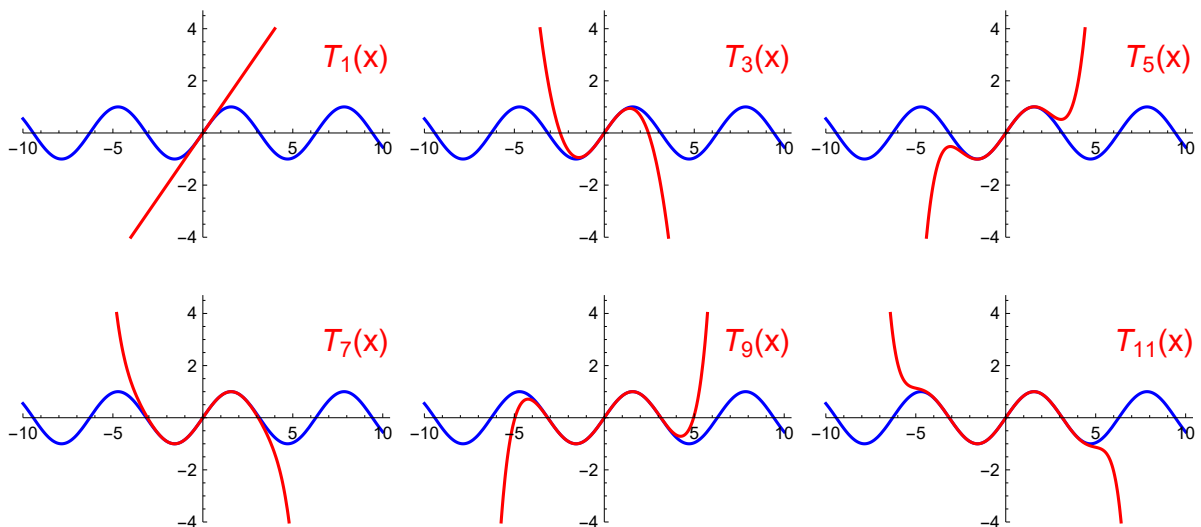


(2) Let $f(x) = \sin x$ and $x_0 = 0$.

$f(x) = \sin x, f'(x) = \cos x, f''(x) = -\sin x, f'''(x) = -\cos x, f^{(4)}(x) = f(x) = \sin x$
 $f(0) = 0, f'(0) = 1, f''(0) = 0, f'''(0) = -1, f^{(4)}(0) = f(0) = 0$

From here it is repeated periodically. The derivatives are uniformly bounded on \mathbb{R}

$$\begin{aligned} \implies \cos x &= f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(k)}(0)}{k!}x^k + \dots = \\ &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \end{aligned}$$



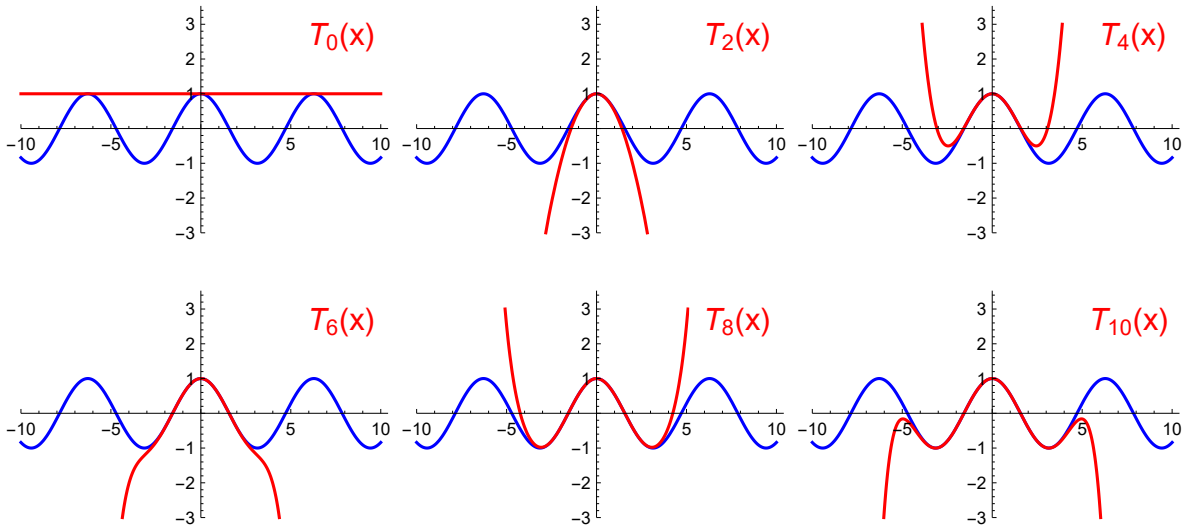
(3) Let $f(x) = \cos x$ and $x_0 = 0$.

$$f(x) = \cos x, \quad f'(x) = -\sin x, \quad f''(x) = -\cos x, \quad f'''(x) = \sin x, \quad f^{(4)}(x) = f(x) = \cos x$$

$$f(0) = 1, \quad f'(0) = 0, \quad f''(0) = -1, \quad f'''(0) = 0, \quad f^{(4)}(0) = f(0) = 1$$

From here it is repeated periodically. The derivatives are uniformly bounded on \mathbb{R}

$$\begin{aligned} \Rightarrow \cos x &= f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(k)}(0)}{k!}x^k + \dots = \\ &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \end{aligned}$$



(4), (5)

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots \quad (x \in \mathbb{R}) \quad \Rightarrow \quad \sinh x = \frac{e^x - e^{-x}}{2} = \frac{1}{2} \sum_{n=0}^{\infty} \frac{x^n - (-x)^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} x^{2n+1}$$

$$e^{-x} = \sum_{n=0}^{\infty} \frac{1}{n!} (-x)^n = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \dots \quad (x \in \mathbb{R}) \quad \cosh x = \frac{e^x + e^{-x}}{2} = \frac{1}{2} \sum_{n=0}^{\infty} \frac{x^n + (-x)^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{(2n)!} x^{2n}$$

Remark. The Taylor series of $f(x) = e^x$ at x_0 is $e^x = e^{x_0} e^{x-x_0} = \sum_{n=0}^{\infty} \frac{e^{x_0}}{n!} (x-x_0)^n$

Examples. (1) The Taylor series of $f(x) = \sin x \cdot \cos x$ at $x_0 = 0$ is

$$f(x) = \sin x \cdot \cos x = \frac{1}{2} \sin 2x = \frac{1}{2} \cdot \left(2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \frac{(2x)^7}{7!} + \dots \right) \quad (x \in \mathbb{R})$$

(2) The Taylor series of $f(x) = e^{-x^2}$ at $x_0 = 0$ is $e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} + \dots \quad (x \in \mathbb{R})$

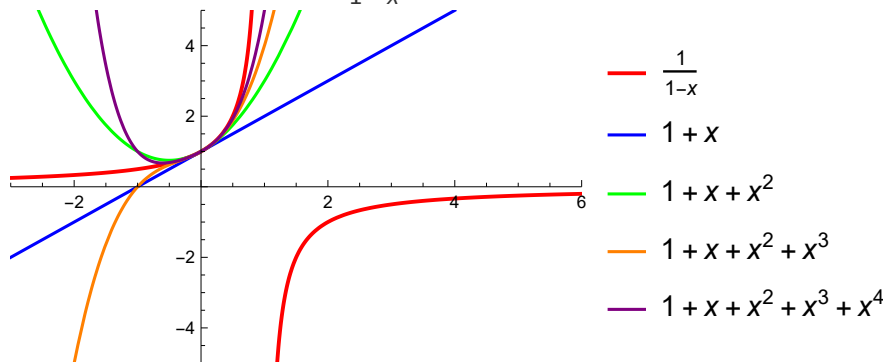
Example. It is known that $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ if $|x| < 1$ (sum of a geometric series).

Find the Taylor series of $f(x) = \frac{1}{1-x}$ at $x_0 = 0$.

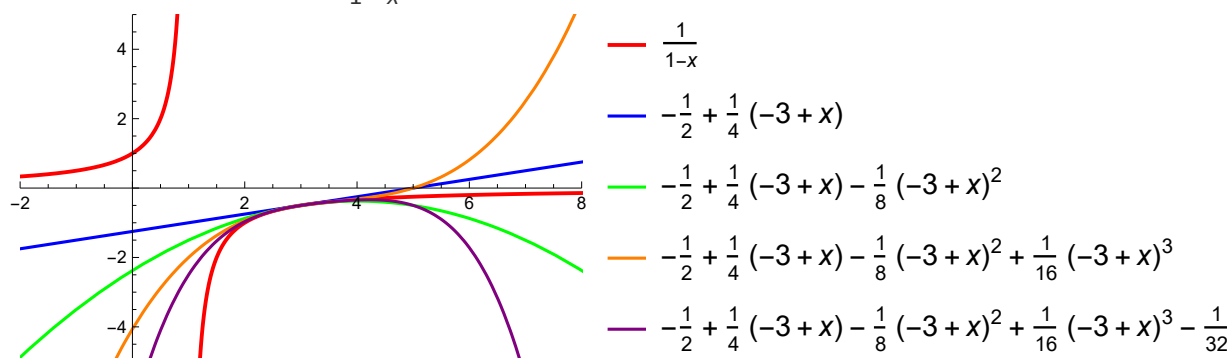
$$f'(x) = \frac{1}{(1-x)^2}, \quad f''(x) = \frac{2}{(1-x)^3}, \quad f'''(x) = \frac{3!}{(1-x)^4}, \quad \dots, \quad f^{(n)}(x) = \frac{n!}{(1-x)^{n+1}} \Rightarrow f^{(n)}(0) = n!$$

$$\Rightarrow T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} x^n \Rightarrow \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \text{if } |x| < 1.$$

Taylor polynomials of $f(x) = \frac{1}{1-x}$ with center $x_0 = 0$:



Taylor polynomials of $f(x) = \frac{1}{1-x}$ with center $x_0 = 3$:



Term by term differentiation and integration

Theorem. If $|x - x_0| < R$ where R is the radius of convergence of the

$$\text{Taylor series } f(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n, \text{ then } f'(x) = \sum_{n=0}^{\infty} n a_n(x - x_0)^{n-1}$$

$$\text{and if } f'(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n, \text{ then } f(x) = f(x_0) + \sum_{n=0}^{\infty} a_n \frac{(x - x_0)^{n+1}}{n+1}.$$

Example. $(e^x)' = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots\right)' = 0 + 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots = e^x$

$$(\sin x)' = \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} + \dots\right)' = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} + \dots = \cos x$$

Statement. $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$ if $x \in [-1, 1]$.

Proof. $f(x) = \arctan x \Rightarrow f'(x) = \frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = 1 - x^2 + x^4 - x^6 + \dots = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$

$f'(x)$ is the sum of a geometric series with ratio $q = -x^2$.

It is convergent $\Leftrightarrow |q| = |-x^2| = |x|^2 < 1 \Rightarrow R = 1$

The interval of convergent for f' is $(-1, 1)$.

$$\Rightarrow f(x) = \arctan x = f(0) + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} + \dots$$

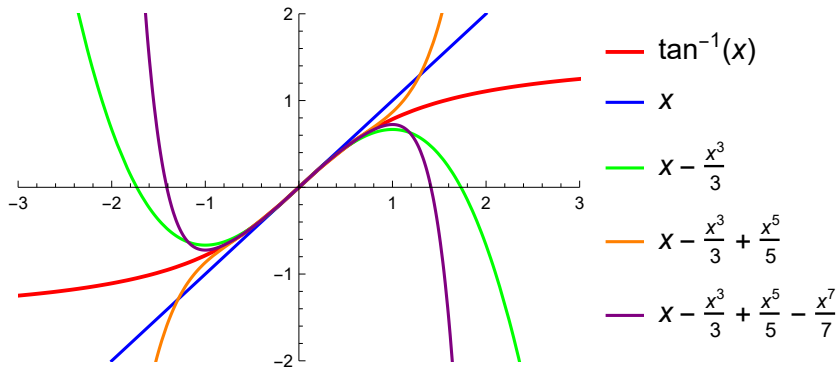
The radius of convergence doesn't change but the endpoints of the interval of convergence can change. Here both endpoints change.

$$\text{If } x = 1 \text{ then } f(1) = \arctg 1 = \frac{\pi}{4} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \dots,$$

this is a Leibniz type series, so it is convergent.

If $x = -1$ then $f(-1) = -f(1)$, so the series is also convergent.

The interval of convergence for f is $[-1, 1]$.



Statement. $\ln(1+x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$ if $x \in [-1, 1]$.

Proof. $f(x) = \ln(1+x) \implies f'(x) = \frac{1}{1+x} = \frac{1}{1-(-x)} = (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n$

$f'(x)$ is the sum a geometric series with ratio $q = -x$.

It is convergent $\iff |q| = |-x| = |x| < 1 \implies R = 1$.

The interval of convergent for f' is $(-1, 1)$.

$$\implies f(x) = \ln(1+x) = f(0) + \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + \dots$$

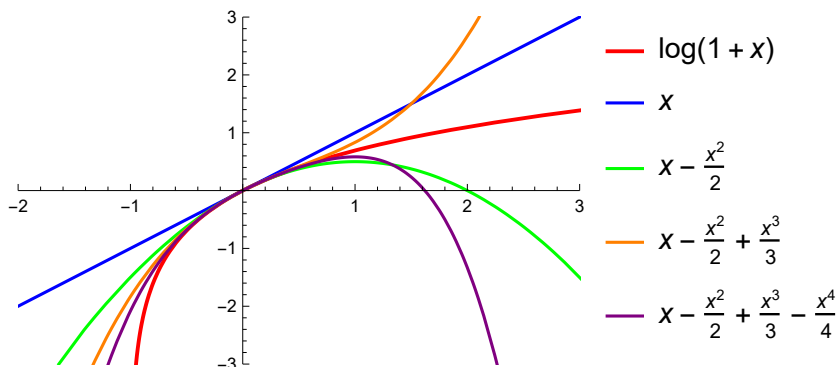
The radius of convergence doesn't change but the endpoints of the interval of convergence can change. Here the right endpoint changes.

If $x = -1$ then $f(x)$ is not defined. If $x = 1$ then

$$f(1) = \ln 2 = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n+1} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \dots, \text{ this is a Leibniz type series,}$$

so it is convergent.

The interval of convergence for f is: $(-1, 1]$.



Binomial series

Example. Find the k th Taylor polynomial of the function $f(x) = (1+x)^\alpha$ (where $\alpha \in \mathbb{R}$) about $x_0 = 0$.

$$\begin{aligned} f(x) &= (1+x)^\alpha && \Rightarrow f(0) = 1 \\ f'(x) &= \alpha(1+x)^{\alpha-1} && \Rightarrow f'(0) = \alpha \\ f''(x) &= \alpha(\alpha-1)(1+x)^{\alpha-2} && \Rightarrow f''(0) = \alpha(\alpha-1) \\ &\dots && \\ f^{(k)}(x) &= \alpha(\alpha-1)(\alpha-k+1)(1+x)^{\alpha-k} && \Rightarrow f^{(k)}(0) = \alpha(\alpha-1)\dots(\alpha-k+1) \\ \Rightarrow T_k(x) &= 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!} x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!} x^3 \dots + \frac{\alpha(\alpha-1)\dots(\alpha-k+1)}{k!} x^k. \end{aligned}$$

Remark. If $\alpha = n \in \mathbb{N}$ then by the binomial theorem

$$(1+x)^n = \sum_{k=0}^n \binom{n}{k} x^k \quad \text{where} \quad \binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)\dots(n-k+1)}{k!}.$$

Definition (Generalized binomial coefficient). If $\alpha \in \mathbb{R}$ and $k \in \mathbb{R}$ then

$$\binom{\alpha}{k} := \frac{\alpha(\alpha-1)\dots(\alpha-k+1)}{k!} \quad \text{and} \quad \binom{\alpha}{0} := 1$$

Remark. The Taylor series of $f(x) = (1+x)^\alpha$ (where $\alpha \in \mathbb{R}$) is $T(x) = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k$.

It is called a **binomial series**.

Theorem. The radius of convergence of the binomial series $\sum_{k=0}^{\infty} \binom{\alpha}{k} x^k$ is $R = 1$.

Proof. $a_k = \binom{\alpha}{k} = \frac{\alpha(\alpha-1)\dots(\alpha-k+1)}{k!}$

$$a_{k+1} = \binom{\alpha}{k+1} = \frac{\alpha(\alpha-1)\dots(\alpha-k+1)(\alpha-k)}{(k+1)!} = \binom{\alpha}{k} \frac{\alpha-k}{k+1}$$

By the ratio test:

$$\lim_{k \rightarrow \infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k \rightarrow \infty} \left| \frac{\alpha-k}{k+1} \right| = 1 = \frac{1}{R} \Rightarrow R = 1$$

Theorem (Sum of a binomial series).

$$(1+x)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k, \quad \text{for all } x \in (-1, 1), \quad \alpha \in \mathbb{R}.$$

Proof. The derivatives of $f(x)$ are not uniformly bounded, so we prove the equality of

$$f(x) = (1+x)^\alpha \quad \text{and} \quad T(x) = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k \quad \text{in the following way.}$$

$$\text{We show that for all } x \in (-1, 1): \left(\frac{T(x)}{f(x)} \right)' \equiv 0 \Rightarrow \frac{T(x)}{f(x)} \equiv \text{constant.}$$

$$\text{Since } f(0) = T(0) = 1 \text{ then } \frac{T(x)}{f(x)} = \frac{T(0)}{f(0)} = 1 \Rightarrow T(x) = f(x) \text{ if } x \in (-1, 1).$$

The derivative is

$$\left(\frac{T(x)}{f(x)}\right)' = \frac{T' \cdot f - T \cdot f'}{f^2} = \frac{T' \cdot (1+x)^\alpha - T \cdot \alpha(1+x)^{\alpha-1}}{f^2} = \frac{(1+x)^{\alpha-1}}{(1+x)^{2\alpha}} ((1+x)T' - \alpha \cdot T)$$

\Rightarrow it is enough to show that $(1+x)T' - \alpha \cdot T \equiv 0$.

For this we use the power series of T and T' .

$$\begin{aligned} T(x) &= 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!} x^2 + \dots + \binom{\alpha}{k} x^k + \dots \\ \alpha T(x) &= \alpha + \alpha^2 x + \frac{\alpha^2(\alpha-1)}{2!} x^2 + \dots + \alpha \binom{\alpha}{k} x^k + \dots \\ T'(x) &= \alpha + \frac{\alpha(\alpha-1)}{1!} x + \frac{\alpha(\alpha-1)(\alpha-2)}{2!} x^2 + \dots + k \binom{\alpha}{k} x^{k-1} + (k+1) \binom{\alpha}{k+1} x^k + \dots \\ x T'(x) &= \alpha x + \frac{\alpha(\alpha-1)}{1!} x^2 + \dots + k \binom{\alpha}{k} x^k + \dots \end{aligned}$$

The above expression as a power series is

$$(1+x)T' - \alpha \cdot T = \sum_{k=0}^{\infty} \left((k+1) \binom{\alpha}{k+1} + k \binom{\alpha}{k} - \alpha \binom{\alpha}{k} \right) x^k$$

where the coefficient of x^k for all $k \in \mathbb{N}$ is

$$(k+1) \binom{\alpha}{k+1} + k \binom{\alpha}{k} - \alpha \binom{\alpha}{k} = \binom{\alpha}{k} \left((k+1) \frac{\alpha-k}{k+1} + k - \alpha \right) = 0.$$

Thus we proved that for all $x \in (-1, 1)$

$$\begin{aligned} (1+x)T'(x) - \alpha \cdot T(x) \equiv 0 &\Rightarrow \left(\frac{T}{f}\right)' \equiv 0 \Rightarrow \frac{T(x)}{f(x)} = \frac{T(0)}{f(0)} = 1 \\ &\Rightarrow T(x) = f(x) \text{ if } x \in (-1, 1). \end{aligned}$$

Example. If $f(x) = \arcsin x$, then $f'(x) = \frac{1}{\sqrt{1-x^2}} = (1+(-x^2))^{-\frac{1}{2}} = \sum_{k=0}^{\infty} \binom{-1/2}{k} (-1)^k x^{2k}$ if $|x| < 1$

$$\Rightarrow \arcsin x = \sum_{k=0}^{\infty} \binom{-1/2}{k} \frac{(-1)^k x^{2k+1}}{2k+1} \text{ if } |x| < 1.$$

Remark. Taylor's formula \Rightarrow

$(1+x)^\alpha = T_n(x) + R_n(x)$, where

$$T_n(x) = \sum_{k=0}^n \binom{\alpha}{k} x^k \text{ and } R_n(x) = \binom{\alpha}{n+1} (1+\xi)^{\alpha-n-1} x^{n+1}, \text{ where } 0 < \xi < x \text{ or } x < \xi < 0.$$

Exercises

Exercise 1. Estimate the value of $\sqrt{2}$ by the Taylor polynomial of order 2 of $f(x) = \sqrt{1+x}$ at center $x_0 = 0$. Give an upper bound for the error for the error of the approximation.

Solution. The derivatives and the substitution values:

$$\begin{aligned} f(x) &= \sqrt{1+x} = (1+x)^{\frac{1}{2}}, \quad f(0) = 1 \\ f'(x) &= \frac{1}{2} (1+x)^{-\frac{1}{2}} = \frac{1}{2\sqrt{1+x}}, \quad f'(0) = \frac{1}{2} \end{aligned}$$

$$f''(x) = -\frac{1}{4} (1+x)^{-\frac{3}{2}} = -\frac{1}{4(1+x)^{3/2}}, \quad f''(0) = -\frac{1}{4}$$

$$f'''(x) = \frac{3}{8} (1+x)^{-\frac{5}{2}} = \frac{3}{8(1+x)^{5/2}}$$

The Taylor polynomial of order 2:

$$T_2(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 = 1 + \frac{1}{2}x - \frac{1}{4 \cdot 2!}x^2$$

$$f(x) \approx T_2(x), \text{ that is, } \sqrt{1+x} \approx 1 + \frac{1}{2}x - \frac{1}{4 \cdot 2!}x^2$$

$$\text{If } x=1 \text{ then } \sqrt{2} \approx T_2(1) = 1 + \frac{1}{2} - \frac{1}{4 \cdot 2!} = 1.375$$

$$\text{Lagrange remainder term: } R_2(x) = \frac{f^{(3)}(\xi)}{3!}(x-x_0)^3, \text{ where } x_0=0, x=1, 0 < \xi < 1$$

The value of ξ is not known so we can only estimate the error.

$$|E| = |R_2(x)| = \left| \frac{3}{8(1+\xi)^{5/2}} \cdot \frac{1}{3!} (1-0)^3 \right| = \frac{1}{16(1+\xi)^{5/2}} < \frac{1}{16(1+0)^{5/2}} = \frac{1}{16} = 0.0625$$

Remark: $\sqrt{2} \approx 1.414213562 \dots$

The approximation is $\sqrt{2} \approx T_2(1) = 1.375$

$$1.4142 - 1.375 = 0.0392$$

Remark: $f(x) = \sqrt{1+x} = (1+x)^{\frac{1}{2}}$

$$\Rightarrow T_2(x) = \sum_{k=0}^2 \binom{1/2}{k} x^k = 1 + \frac{1}{2}x + \frac{\frac{1}{2} \cdot \left(-\frac{1}{2}\right)}{2!} x^2 = 1 + \frac{1}{2}x - \frac{1}{8}x^2$$

$$R_2(x) = \binom{1/2}{3} (1+\xi)^{\frac{1}{2}-3} x^3 = \frac{\frac{1}{2} \cdot \left(-\frac{1}{2}\right) \cdot \left(-\frac{3}{2}\right)}{3!} \cdot \frac{1}{(1+\xi)^{\frac{5}{2}}} \cdot x^3 = \frac{3}{8 \cdot 3!} \cdot \frac{1}{(1+\xi)^{\frac{5}{2}}} \cdot x^3$$

Exercise 2. We estimate the value of $\ln(1.1)$ by the Taylor polynomial of order n of $f(x) = \ln(1+x)$ at center $x_0 = 0$. Find n if the error for the approximation is less than 10^{-8} .

Solution. The first few derivatives are

$$f(x) = \ln(1+x), \quad f(0) = 0 \qquad f'''(x) = \frac{2}{(1+x)^3}, \quad f'''(0) = 2 = 2!$$

$$f'(x) = \frac{1}{1+x}, \quad f'(0) = 1 = 0! \qquad f^{(4)}(x) = -\frac{6}{(1+x)^4}, \quad f^{(4)}(0) = -6 = -3!$$

$$f''(x) = -\frac{1}{(1+x)^2}, \quad f''(0) = -1 = -1! \qquad f^{(5)}(x) = \frac{24}{(1+x)^5}, \quad f^{(5)}(0) = 24 = 4! \text{ etc.}$$

$$\Rightarrow f^{(n)}(x) = (-1)^{n-1} \frac{(n-1)!}{(1+x)^n} \Rightarrow f^{(n)}(0) = (-1)^{n-1} \cdot (n-1)!$$

The Taylor polynomial at $x_0 = 0$:

$$T_n(x) = 0 + 1 \cdot (x-0) - \frac{1}{2!} (x-0)^2 + \frac{2}{3!} (x-0)^3 + \frac{-6}{4!} (x-0)^4 + \dots = x - \frac{1}{2} x^2 + \frac{1}{3} x^3 - \frac{1}{4} x^4 + \dots$$

$$\text{If } x = 0.1: \ln(1.1) \approx 0.1 - \frac{1}{2} \cdot 0.1^2 + \frac{1}{3} \cdot 0.1^3 - \frac{1}{4} \cdot 0.1^4 + \dots$$

$$\text{Lagrange remainder term: } R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} (x-x_0)^{n+1}, \text{ where } x_0 = 0, x = 0.1, 0 < \xi < 0.1.$$

Taylor's theorem: $f(x) = T_n(x) + R_n(x)$

$$\begin{aligned} \text{The error: } |E| &= |f(x) - T_n(x)| = |R_n(x)| = \left| \frac{(-1)^n \cdot n!}{(1+\xi)^{n+1} \cdot (n+1)!} (x-x_0)^{n+1} \right| = \\ &= \left| \frac{(-1)^n \cdot n!}{(1+\xi)^{n+1} \cdot (n+1)!} (0.1-0)^{n+1} \right| = \frac{1}{(n+1)(1+\xi)^{n+1}} \cdot 0.1^{n+1} < \frac{1}{(n+1)(1+0)^{n+1}} \cdot 0.1^{n+1} = \\ &= \frac{0.1^{n+1}}{(n+1)} < 10^{-8} \implies n \geq 7 \end{aligned}$$

Comparison of the numerical values:

$$\ln(1.1) \approx 0.09531017980 \dots$$

The approximation is

$$\ln(1.1) \approx T_7(0.1) = 0.1 - \frac{1}{2} \cdot 0.1^2 + \frac{1}{3} \cdot 0.1^3 - \frac{1}{4} \cdot 0.1^4 + \frac{1}{5} \cdot 0.1^5 - \frac{1}{6} \cdot 0.1^6 + \frac{1}{7} \cdot 0.1^7 \approx 0.09531018095 \dots$$

\implies the first 7 digits are accurate.