Calculus 1 - 10

Solutions - Basic topological concepts

Are the following statements true or false?

1. Let $H \subset \mathbb{R}$.
   a) If $x \in H$, then x is an interior point of H.
   b) If $x \in H$, then x cannot be a boundary point of H.

Solution.
a) False, for example, if $H = [0, 1]$ or $H = \{0, 1\}$ then $x = 0$ is a boundary point of H.
b) False, see the examples in part a).

2. Let $H \subset \mathbb{R}$. If $x \notin H$, then x cannot be
   a) an interior point of H;
   b) a boundary point of H;
   c) a limit point of H;
   d) an isolated limit point of H.

Solution.
a) True, since $\operatorname{int} H \subset H$.
b) False, for example if $H = (0, 1)$ then $x = 0$ is a boundary point of H.
c) False, see the previous example.
d) True, if x is an isolated point of H, then $x \in H$.

3. a) If x is an interior point of $H \subset \mathbb{R}$, then x is a limit point of H.
   b) If x is a boundary point of $H \subset \mathbb{R}$, then x is a limit point of H.

Solution.
a) True. If x is an interior point of H, then there exists $R > 0$ such that $B(x, R) = (x - R, x + R) \subset H$.
Then for all $r > 0$: $(B(x, r) \setminus \{x\}) \cap B(x, R) \neq \emptyset$, that is, any neighbourhood of x contains a point in $B(x, R)$ that is distinct from x. Since $B(x, R) \subset H$ then x is a limit point of H.
b) False. For example, if $H = \{1, 2, 3\}$, then $x = 1$ is an isolated point of H, which is boundary point but not a limit point of H.

4. a) If x is a limit point of $H \subset \mathbb{R}$, then x is a boundary point of H.
   b) If x is a limit point of $H \subset \mathbb{R}$, then x is an interior point or a boundary point of H.

Solution.
a) False. For example, if $H = (0, 2)$, then $x = 1$ is both an interior point and a limit point of H, but not a boundary point.
b) True. Assume that x is a limit point of H, which means that any interval $(x - r, x + r)$ contains a point in H that is distinct from x. There are two possibilities:
Case 1. If there exists an interval $(x - r, x + r)$ that doesn't contain a point in $\operatorname{ext} H$, then $(x - r, x + r) \subset H$, so x is an interior point of H.
Case 2. If every interval $(x - r, x + r)$ contains a point in $\operatorname{ext} H$, then x is a boundary point of H.

5. a) If x is an isolated point of $H \subset \mathbb{R}$, then x is boundary point of H.
b) If $H \subset \mathbb{R}$, $x \in H$ and x is not an isolated point of H, then x is an interior point of H.

Solution.
a) True. If x is an isolated point of H, then there exists $R > 0$ such that $(x - R, x + R) \cap H = \{x\}$. Then for all $r > 0$ the interval $(x - r, x + r)$ contains a point in H, since it contains x. Moreover, the interval $(x - r, x + r)$ also contains infinitely many points not in H, these are the points in $(x - R, x + R) \cap (x - r, x + r) \setminus \{x\}$. That is, any interval $(x - r, x + r)$ intersects both H and $\mathbb{R} \setminus H$. Therefore, x is a boundary point of H.
b) False. For example, if $H = [0, 1]$ and $x = 0$, then $x \in H$ and x is not an isolated point of H

6. a) If $x \in \mathbb{R}$ has a neighbourhood that contains infinitely many points of $H \subset \mathbb{R}$, then x is a limit point of H.
b) If every neighbourhood of $x \in \mathbb{R}$ contains infinitely many points of $H \subset \mathbb{R}$, then x is a limit point of H.

Solution.
a) False. For example let $x = 0$ and $H = [1, 2]$. Then with $r = 3$ the interval $B(x, r) = (-3, 3)$ is a neighbourhood of $x = 0$, which contains infinitely many points of H, but $x = 0$ is not a limit point of H.
b) True, this is equivalent with the definition of the limit point.

7. There exists a set $H \subset \mathbb{R}$ which has
a) no interior points; b) no boundary points;
c) no limit points; d) no isolated points.

Solution.
a) True. For example $H = \{1, 2, 3\}$ or $H = \mathbb{Z}$ or $H = [0, 1] \cap \mathbb{Q}$.
b) True. $H = \emptyset$ or $H = \mathbb{R}$.
c) True. For example $H = \{1, 2, 3\}$ or $H = \mathbb{Z}$.
d) True. For example $H = (0, 1)$.

8. There exists a set $H \subset \mathbb{R}$ such that all points of H are
a) interior points; b) boundary points;
c) limit points; d) isolated points.

Solution.
a) True. For example $H = (0, 1)$ or $H = \mathbb{R} \setminus \{1, 2, 3\}$.
b) True. For example $H = \{1, 2, 3\}$ or $H = [0, 1] \cap \mathbb{Q}$ or $H = \left\{\frac{1}{n} : n \in \mathbb{N}^+\right\}$.
c) True. For example $H = (0, 1)$ or $H = [0, 1]$ or $H = \mathbb{R} \setminus \{1, 2, 3\}$ or $H = [0, 1] \cap \mathbb{Q}$.
d) True. For example $H = \{1, 2, 3\}$ or $H = \mathbb{Z}$ or $H = \left\{\frac{1}{n} : n \in \mathbb{N}^+\right\}$.

9. There exists a set $H \subset \mathbb{R}$ which has
a) exactly one interior point;
b) exactly one limit point;
c) exactly one boundary point.

Solution.

a) False. If x is an interior point of H, then there exists $r > 0$ such that $B(x, r) = (x - r, x + r) \subset H$. Then all points of the open interval $(x - r, x + r)$ are also interior points of H.

b) True. Let H be range of any convergent sequence, for example $H = \left\{\frac{1}{n} : n \in \mathbb{N}^+\right\}$ or $H = \left\{\left(1 + \frac{1}{n}\right)^n : n \in \mathbb{N}\right\}$.

c) True. For example $H = \mathbb{R} \setminus \{0\}$.

10. There exists a set $H \subset \mathbb{R}$ which is equal to the
    a) set of its interior points;
    b) set of its limit points;
    c) set of its boundary points.

Solution.

a) True. In this case H is an open set, for example $H = (0, 1)$.

b) True. In this case H is a closed set, for example $H = [0, 1]$.

c) True. In this case all points of H are isolated points, for example $H = \{1, 2, 3\}$ or $H = \mathbb{Z}$.

11. a) The set $H = [0, 1] \cap \mathbb{Q}$ is open.
    b) The set $H = [0, 1] \cap \mathbb{Q}$ is closed.
    c) The set $H = \left\{\frac{1}{n} : n \in \mathbb{N}^+\right\}$ is closed.

Reminder:

- H is open $\iff \forall x \in H \quad \exists r > 0$ such that $(x - r, x + r) \subset H$
- H is not open $\iff \exists x \in H \quad \forall r > 0$ such that $(x - r, x + r)$ is not a subset of H
- Theorem: $H \subset \mathbb{R}$ is closed $\iff$ H contains all of its limit points.

Solution.

a) False. Consider the point $x = \frac{1}{2} \in H$. Then, since for all $r > 0$ the interval $(x - r, x + r)$ contains irrational numbers in $[0, 1]$, then $(x - r, x + r)$ cannot be a subset of H, so H is not open.

b) False. Similarly as in part a) it can be proved that the complement of H is not open, so H is not closed.
The complement of H is $\mathbb{R} \setminus H = (-\infty, 0) \cup (1, \infty) \cup ((0, 1) \setminus \mathbb{Q})$. If x is an irrational number between 0 and 1 then for all $r > 0$ the interval $(x - r, x + r)$ contains rational numbers in $(0, 1)$, so $(x - r, x + r)$ cannot be a subset of $\mathbb{R} \setminus H$.
Or:
H is not closed, since H doesn't contain the irrational numbers in $[0, 1]$ which are limit points of H.

c) False, since $x = 0$ is a limit point of H, but $x \notin H$.

12. a) If the set $H \subset \mathbb{R}$ is open, then every point of H is an interior point.
    b) If the set $H \subset \mathbb{R}$ is closed, then every point of H is a boundary point.

Solution.

a) True by the definition of the open set and the interior point.

b) False. For example $H = [0, 1]$ is closed, but the points if $(0, 1)$ are interior points of H.

13. a) If the set $H \subset \mathbb{R}$ is closed, then every point of H is a limit point.
    b) If every point of the set $H \subset \mathbb{R}$ is a limit point, then H is closed.

Solution.

a) False. For example $H = \{1, 2, 3\}$ is closed, but it has no limit points.

b) False. For example every points of $H = (0, 1)$ is a limit point, but H is not closed.

14. a) If the set $H \subset \mathbb{R}$ is closed, then it contains all of its limit points.
    b) If the set $H \subset \mathbb{R}$ contains all of its limit points, then it is closed.

Solution. Both a) and b) are true by the following theorem:
$H \subset \mathbb{R}$ is closed if and only if it contains all of its limit points.

15. a) If the set $H \subset \mathbb{R}$ has finitely many points, then it has no limit points.
    b) If the set $H \subset \mathbb{R}$ has infinitely many points, then it has at least one limit point.
    c) If the set $H \subset \mathbb{R}$ is bounded and has infinitely many points, then it has at least one limit point.

Solution.

a) True. If $x \in \mathbb{R}$ is a limit point of H, then any neighbourhood of x must contain infinitely many elements of H. However, since H has only finitely many points, then it is not possible.

b) False. For example $H = \mathbb{Z}$ has infinitely many points, but it has no limit points.

c) True. Let (a_n) be a sequence in H. Then (a_n) is bounded, so by the Bolzano-Weierstrass theorem it has a convergent subsequence, let us denote its limit by A. Then A is a limit point of H, since any neighbourhood of A contains infinitely many points in H.