

Calculus 1 - 05

Number sequences, part 2.

Binomial theorem

Binomial coefficients: $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ where $k! = 1 \cdot 2 \cdot \dots \cdot k$ and $0! = 1$.

Meaning: the number of subsets with k elements of a set with n elements.

Binomial theorem: $(a+b)^n = (a+b)(a+b)\dots(a+b) = \sum_{k=0}^n \binom{n}{k} a^k b^{n-k}$.

Orders of magnitudes

Definition: Suppose that $a_n \xrightarrow{n \rightarrow \infty} \infty$ and $b_n \xrightarrow{n \rightarrow \infty} \infty$. Then the order of magnitude of (a_n) is smaller than the order of magnitude of (b_n) if $\frac{a_n}{b_n} \xrightarrow{n \rightarrow \infty} 0$.

Notation: $a_n \ll b_n$.

Theorem: $n^n \gg n! \gg a^n \gg n^k \gg n^{\frac{1}{k}} \gg \log n$, where $a > 1$ and $k \in \mathbb{N}^+$. That is,

a) $\lim_{n \rightarrow \infty} \frac{n^n}{n!} = \infty$ **b)** $\lim_{n \rightarrow \infty} \frac{n!}{a^n} = \infty$, where $a > 1$ **c)** $\lim_{n \rightarrow \infty} \frac{a^n}{n} = \infty$, where $a > 1$

d) $\lim_{n \rightarrow \infty} \frac{a^n}{n^k} = \infty$, where $a > 1$ and $k \in \mathbb{N}^+$ **e)** $\lim_{n \rightarrow \infty} \frac{n}{\log_2 n} = \infty$

Some proofs. a) $\frac{n^n}{n!} = \frac{n}{1} \cdot \frac{n}{2} \cdot \frac{n}{3} \cdot \dots \cdot \frac{n}{n-2} \cdot \frac{n}{n-1} \cdot \frac{n}{n} \geq n \cdot 1 \cdot 1 \cdot \dots \cdot 1 \cdot 1 \cdot 1 = n \rightarrow \infty \implies \frac{n^n}{n!} \rightarrow \infty$

b) For example, if $a = 2$, then $\frac{n!}{2^n} = \frac{n}{2} \cdot \frac{n-1}{2} \cdot \frac{n-2}{2} \cdot \dots \cdot \frac{3}{2} \cdot \frac{2}{2} \cdot \frac{1}{2} \geq \frac{n}{2} \cdot 1 \cdot 1 \cdot \dots \cdot 1 \cdot 1 \cdot \frac{1}{2} = \frac{n}{4} \rightarrow \infty \implies \frac{n!}{2^n} \rightarrow \infty$

In general, if $a > 1$, then $\frac{n!}{a^n} = \frac{n}{a} \cdot \frac{n-1}{a} \cdot \dots \cdot \frac{[a]+1}{a} \cdot \frac{[a]}{a} \cdot \dots \cdot \frac{1}{a} \geq \frac{n}{a} \cdot 1 \cdot \dots \cdot 1 \cdot c = \frac{c}{a} \cdot n \rightarrow \infty \implies \frac{n!}{a^n} \rightarrow \infty$,

where $c = \frac{[a]}{a} \cdot \dots \cdot \frac{1}{a}$.

c) We will prove that $\lim_{n \rightarrow \infty} \frac{a^n}{n} = \infty$, where $a = 1 + \delta$ and $\delta > 0$. By the binomial theorem,

$$(1 + \delta)^n = \sum_{k=0}^n \binom{n}{k} \delta^k = \binom{n}{0} \delta^0 + \binom{n}{1} \delta^1 + \binom{n}{2} \delta^2 + \dots + \binom{n}{n} \delta^n \geq \binom{n}{2} \delta^2, \text{ so}$$

$$\frac{(1 + \delta)^n}{n} \geq \frac{\binom{n}{2} \delta^2}{n} = \frac{n(n-1)}{2n} \delta^2 = \frac{n-1}{2} \delta^2 \rightarrow \infty \implies \frac{a^n}{n} \rightarrow \infty, \text{ where } a > 1.$$

d) We will prove that $\lim_{n \rightarrow \infty} \frac{a^n}{n^k} = \infty$, where $a > 1$ and $k \in \mathbb{N}^+$. This is a consequence of case c),

$$\text{since if } a > 1 \text{ then } \sqrt[k]{a} > 1 \text{ and } \frac{a^n}{n^k} = \left(\frac{(\sqrt[k]{a})^n}{n} \right)^k.$$

e) Let $a_n = \frac{n}{\log_2 n}$. It can be shown that (a_n) is monotonic increasing (we can prove this later)

$$\text{and } a_{2^k} = \frac{2^k}{\log_2 2^k} = \frac{2^k}{k} \rightarrow \infty. \text{ From these two properties it follows that } a_n \rightarrow \infty.$$

Example:
$$\frac{n^2 - 3^n}{n! + n^4} = \frac{3^n}{n!} \cdot \frac{\frac{n^2}{3^2} - 1}{1 + \frac{n^4}{n!}} \xrightarrow{n \rightarrow \infty} 0 \cdot \frac{0 - 1}{1 + 0} = 0.$$

Theorem. $\lim_{n \rightarrow \infty} n^k a^n = 0$, if $|a| < 1$ and $k \in \mathbb{N}^+$.

1st proof. It is a consequence of the following statements:

- a) If $a_n \xrightarrow{n \rightarrow \infty} \infty$ then $\frac{1}{a_n} \xrightarrow{n \rightarrow \infty} 0$.
- b) If $a > 1$ and $k \in \mathbb{N}^+$ then $\frac{a^n}{n^k} \xrightarrow{n \rightarrow \infty} \infty$.
- c) If $|a_n| \xrightarrow{n \rightarrow \infty} 0$ then $a_n \xrightarrow{n \rightarrow \infty} 0$.

2nd proof. It is a consequence of the following statements:

- (i) $\lim_{n \rightarrow \infty} \sqrt[n]{n} = 1$ (see the proof later)
- (ii) If $0 < \lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = L < 1$ then $a_n \xrightarrow{n \rightarrow \infty} 0$.

Proof of (ii): If $L \leq q < 1$ then there exists $N \in \mathbb{N}$ such that for all $n > N$, $\sqrt[n]{|a_n|} < q$.

Then $0 < |a_n| < q^n \rightarrow 0$ so by the Sandwich Theorem $a_n \xrightarrow{n \rightarrow \infty} 0$.

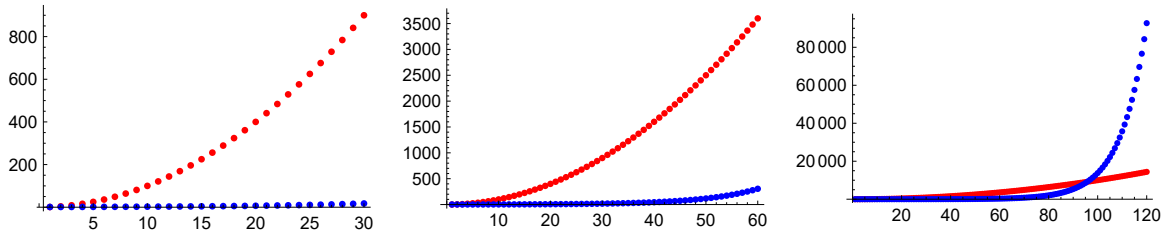
Using this, if $|a| < 1$ then $\sqrt[n]{|n^k a^n|} = (\sqrt[n]{n})^k \cdot |a| \rightarrow 1^k \cdot |a| < 1 \Rightarrow n^k a^n \rightarrow 0$.

Example:
$$\lim_{n \rightarrow \infty} \frac{n^2 + 9^{n+1}}{2n^5 + 3^{2n-1}} = ?$$

Solution:
$$\frac{n^2 + 9^{n+1}}{2n^5 + 3^{2n-1}} = \frac{9^n}{9^n} \cdot \frac{n^2 \left(\frac{1}{9}\right)^n + 9}{2n^5 \left(\frac{1}{9}\right)^n + \frac{1}{3}} \xrightarrow{n \rightarrow \infty} \frac{0 + 9}{0 + \frac{1}{3}} = 27.$$

We used that $\lim_{n \rightarrow \infty} n^k a^n = 0$, if $|a| < 1$. Here $a = \frac{1}{9}$.

The graph of the sequences $a_n = n^2$ and $b_n = 1.1^n$



Subsequences

Definition. Suppose that $(n_k) : \mathbb{N} \rightarrow \mathbb{N}$ is a strictly monotonically increasing sequence of natural numbers. Then we call the sequence (a_{n_k}) a **subsequence** of (a_n) .

Examples: 1) The prime numbers are a subsequence of the positive integers.

2) $b_n = \frac{1}{1+n^2}$ is a subsequence of $a_n = \frac{1}{1+n}$ ($b_n = a_{n^2}$).

3) $c_n = \frac{1}{n!}$ is a subsequence of $a_n = \frac{1}{n}$ ($c_n = a_{n!}$).

Remark. A subsequence can be obtained from a given sequence by deleting some or no elements without changing the order of the remaining elements.

For example, 2, 4, 6, 8, ... is a subsequence of 1, 2, 3, 4, 5, 6, 7, 8, ...,
but 4, 2, 8, 6, ... is not a subsequence of it.

Remark. If (n_k) is a strictly monotonically increasing sequence of natural numbers, then $n_k \xrightarrow{k \rightarrow \infty} \infty$ since $n_k \geq n_1 + k - 1$.

Theorem. $\lim_{n \rightarrow \infty} a_n = A$ if and only for all (a_{n_k}) subsequences $\lim_{k \rightarrow \infty} a_{n_k} = A$.

Proof. 1) Assume that all (a_{n_k}) subsequences tend to the same limit A .

Since (a_{n+1}) is also a subsequence of (a_n) , then $\lim_{n \rightarrow \infty} a_{n+1} = A$,

so for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that if $n > N$ then $|a_{n+1} - A| < \varepsilon$.

Then obviously $|a_n - A| < \varepsilon$ if $n > N + 1$, so $\lim_{n \rightarrow \infty} a_n = A$ also holds.

2) Assume that $\lim_{n \rightarrow \infty} a_n = A$ and let (a_{n_k}) be a subsequence of (a_n) .

Then for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$, such that if $n > N$, then $|a_n - A| < \varepsilon$.

Since $n_k \xrightarrow{k \rightarrow \infty} \infty$, then for the number $N \in \mathbb{N}$ above, there exists $S \in \mathbb{N}$ such that

if $k > S$, then $n_k > N$, so $|a_{n_k} - A| < \varepsilon$, therefore $a_{n_k} \xrightarrow{k \rightarrow \infty} A$.

The Sandwich Theorem and two applications

Theorem (Sandwich Theorem). If $a_n \xrightarrow{n \rightarrow \infty} A \in \mathbb{R}$, $c_n \xrightarrow{n \rightarrow \infty} A \in \mathbb{R}$ and $a_n \leq b_n \leq c_n$ for all $n > N$, then $b_n \xrightarrow{n \rightarrow \infty} A \in \mathbb{R}$

Proof. Let $\varepsilon > 0$ be fixed. Then

there exists $N_1 \in \mathbb{N}$ such that if $n > N_1$ then $A - \varepsilon < a_n < A + \varepsilon$ and

there exists $N_2 \in \mathbb{N}$ such that if $n > N_2$ then $A - \varepsilon < c_n < A + \varepsilon$.

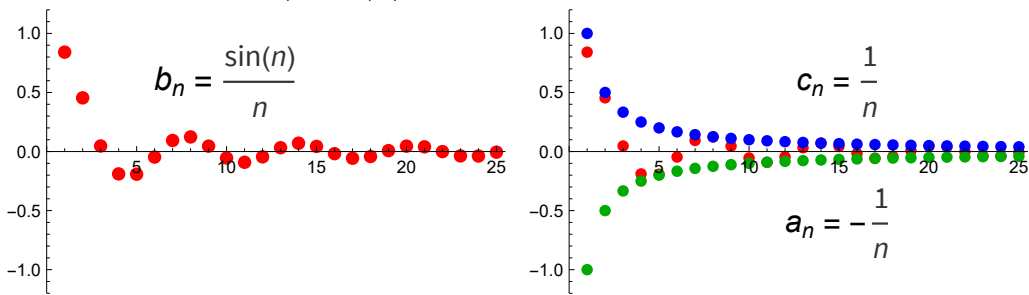
So if $n > \max\{N, N_1, N_2\}$ then

$$A - \varepsilon < a_n \leq b_n \leq c_n < A + \varepsilon \implies |b_n - A| < \varepsilon.$$

Example: Calculate the limit of the sequence $a_n = \frac{\sin(n)}{n}$.

Solution. For all n we have $-1 \leq \sin(n) \leq 1 \implies -\frac{1}{n} \leq \frac{\sin(n)}{n} \leq \frac{1}{n}$.

Since $\lim_{n \rightarrow \infty} \left(-\frac{1}{n}\right) = \lim_{n \rightarrow \infty} \left(\frac{1}{n}\right) = 0$, then by the Sandwich Theorem $\lim_{n \rightarrow \infty} \frac{\sin(n)}{n} = 0$.



Theorem. $\lim_{n \rightarrow \infty} \sqrt[n]{n} = 1$.

1st proof. Apply the AM-GM inequality for $a_1 = \dots = a_{n-2} = 1$, $a_{n-1} = a_n = \sqrt{n}$. Then

$$1 \leq \sqrt[n]{n} = \sqrt[n]{1 \cdot \dots \cdot 1 \cdot \sqrt{n} \cdot \sqrt{n}} \leq \frac{(n-2) + 2\sqrt{n}}{n} \leq 1 + \frac{2}{\sqrt{n}} \rightarrow 1 + 0 - 0 = 1,$$

so by the Sandwich Theorem, $\sqrt[n]{n} \rightarrow 1$.

2nd proof. Since $\sqrt[n]{n} \geq 1$ then we can write $\sqrt[n]{n} = 1 + \delta_n$, where $\delta_n \geq 0$. Then by the binomial theorem, n can be estimated from below:

$$n = (1 + \delta_n)^n = 1 + n\delta_n + \binom{n}{2}\delta_n^2 + \dots + \binom{n}{n}\delta_n^n \geq \binom{n}{2}\delta_n^2 = \frac{n(n-1)}{2}\delta_n^2, \text{ from where}$$

$$0 \leq \delta_n \leq \sqrt{\frac{2}{n-1}} \rightarrow 0, \text{ so by the Sandwich Theorem, } \delta_n \rightarrow 0 \text{ and thus } \sqrt[n]{n} \rightarrow 1.$$

Theorem. If $p > 0$ then $\lim_{n \rightarrow \infty} \sqrt[n]{p} = 1$.

1st proof. Assume that $p \geq 1$ and apply the AM-GM inequality for $a_1 = \dots = a_{n-2} = 1$, $a_{n-1} = a_n = \sqrt{p}$. Then

$$1 \leq \sqrt[n]{p} = \sqrt[n]{1 \cdot \dots \cdot 1 \cdot \sqrt{p} \cdot \sqrt{p}} \leq \frac{(n-2) + 2\sqrt{p}}{n} \leq 1 + \frac{2\sqrt{p}}{n} \rightarrow 1 + 0 = 1,$$

so by the Sandwich Theorem, $\sqrt[n]{p} \rightarrow 1$.

If $0 < p < 1$, then $\frac{1}{p} > 1$, so $\sqrt[n]{p} = \frac{1}{\sqrt[n]{\frac{1}{p}}} \rightarrow 1$.

2nd proof. If $p \geq 1$ then $\sqrt[n]{p} \geq 1$, so we can write $\sqrt[n]{p} = 1 + \delta_n$, where $\delta_n \geq 0$. Then by the binomial theorem, n can be estimated from below:

$$p = (1 + \delta_n)^n = 1 + n \delta_n + \binom{n}{2} \delta_n^2 + \dots + \binom{n}{n} \delta_n^n \geq n \delta_n,$$

from where $0 \leq \delta_n \leq \frac{p}{n} \rightarrow 0$, so by the Sandwich Theorem, $\delta_n \rightarrow 0$ and thus $\sqrt[n]{p} \rightarrow 1$.

The case $0 < p < 1$ is the same as before.

3rd proof. If $p \geq 1$ then $\sqrt[n]{p} \geq 1$, so we can write $\sqrt[n]{p} = 1 + \delta_n$, where $\delta_n \geq 0$. We show that $\delta_n \rightarrow 0$.

$$\text{By the Bernoulli inequality } p = (1 + \delta_n)^n \geq 1 + n \delta_n \implies \frac{p-1}{n} \geq \delta_n > 0.$$

Since $\frac{p-1}{n} \rightarrow 0$ then by the Sandwich Theorem $\delta_n \rightarrow 0$, so $\sqrt[n]{p} \rightarrow 1$.

The case $0 < p < 1$ is the same as before.

Examples

Exercise 1. Calculate the limit of $a_n = \sqrt[3n]{n}$.

1st solution: $a_n = \sqrt[3n]{n} = \sqrt[3n]{\frac{3n}{3}} = \frac{\sqrt[3n]{3n}}{\sqrt[3n]{3}} \rightarrow \frac{1}{1} = 1$. Here we use that $\sqrt[3n]{3n} \rightarrow 1$, since it is a

subsequence of $\sqrt[n]{n}$, and similarly $\sqrt[3n]{3} \rightarrow 1$, since it is a subsequence of $\sqrt[n]{3}$.

2nd solution: $a_n = \sqrt[3n]{n} = \sqrt[3]{\sqrt[n]{n}} \rightarrow \sqrt[3]{1} = 1$.

3rd solution: Since $1 \leq a_n = \sqrt[3n]{n} \leq \sqrt[3n]{3n}$ for all $n \in \mathbb{N}$ and $\sqrt[3n]{3n} \rightarrow 1$ then by the Sandwich Theorem, $a_n \rightarrow 1$.

Exercise 2. Calculate the limit of $a_n = \sqrt[n]{\frac{2n^5 + 5n}{8n^2 - 2}}$.

Solution. Estimating a_n from above and from below:

$$a_n = \sqrt[n]{\frac{2n^5 + 5n}{8n^2 - 2}} \leq \sqrt[n]{\frac{2n^5 + 5n^5}{8n^2 - 2n^2}} = \sqrt[n]{\frac{7n^5}{6n^2}} = \sqrt[n]{\frac{7}{6}} \cdot (\sqrt[n]{n})^3 \rightarrow 1 \cdot 1^3 = 1,$$

$$a_n = \sqrt[n]{\frac{2n^5 + 5n}{8n^2 - 2}} \geq \sqrt[n]{\frac{2n^5 + 0}{8n^2 + 0}} = \sqrt[n]{\frac{2n^5}{8n^2}} = \sqrt[n]{\frac{2}{8}} \cdot (\sqrt[n]{n})^3 \rightarrow 1 \cdot 1^3 = 1,$$

so by the Sandwich Theorem, $a_n \rightarrow 1$.

Exercise 2. Calculate the limit of $a_n = \sqrt[n]{\frac{3^n + 5^n}{2^n + 4^n}}$.

Solution. Estimating a_n from above and from below:

$$a_n = \sqrt[n]{\frac{3^n + 5^n}{2^n + 4^n}} \leq \sqrt[n]{\frac{5^n + 5^n}{0 + 4^n}} = \sqrt[n]{2} \cdot \frac{5}{4} \rightarrow 1 \cdot \frac{5}{4} = \frac{5}{4},$$

$$a_n = \sqrt[n]{\frac{3^n + 5^n}{2^n + 4^n}} \geq \sqrt[n]{\frac{0 + 5^n}{4^n + 4^n}} = \sqrt[n]{\frac{1}{2}} \cdot \frac{5}{4} \rightarrow 1 \cdot \frac{5}{4} = \frac{5}{4},$$

so by the Sandwich Theorem, $a_n \rightarrow \frac{5}{4}$.

Monotonic sequences

Theorem. If (a_n) is monotonically increasing and not bounded above, then $a_n \xrightarrow{n \rightarrow \infty} \infty$.

Proof. Let $P > 0$ be fixed. Since it is not an upper bound, there exists an $N \in \mathbb{N}$ such that $a_N > P$.

By the monotonicity, if $n > N$ then $a_n \geq a_N > P$.

Consequence. If (a_n) is monotonically decreasing and not bounded below, then $a_n \xrightarrow{n \rightarrow \infty} -\infty$.

Theorem. (1) If (a_n) is monotonically increasing and bounded above, then (a_n) is convergent and

$$\lim_{n \rightarrow \infty} a_n = \sup \{a_n : n \in \mathbb{N}\}.$$

(2) If (a_n) is monotonically decreasing and bounded below, then (a_n) is convergent and

$$\lim_{n \rightarrow \infty} a_n = \inf \{a_n : n \in \mathbb{N}\}.$$

Consequence. If (a_n) is monotonic and bounded then (a_n) is convergent.

Proof of part (1).

1. Let $A = \sup \{a_k : k \in \mathbb{N}\}$, then $a_n \leq A$ for all $n \in \mathbb{N}$.

2. Assume indirectly that $\lim_{n \rightarrow \infty} a_n \neq A$. Then there exists $\varepsilon > 0$, such that for all $N \in \mathbb{N}$

there exists $n > N$, such that $a_n \leq A - \varepsilon$.

3. By the monotonicity $a_N \leq a_n$, so $a_N \leq A - \varepsilon$ for all $N \in \mathbb{N}$.

4. However, this is a contradiction, since A is the smallest upper bound of the sequence, so $A - \varepsilon$ is not an upper bound.

Therefore for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that if $n > N$ then $A - \varepsilon < a_n \leq A < A + \varepsilon$, so $\lim_{n \rightarrow \infty} a_n = A$.

Recursive sequences

In many cases, the convergence of recursively given sequences can be investigated by the application of the previous theorem.

Exercise 1. Let $0 < a < 1$ and $b_n = a^n$. Prove that the sequence (b_n) is convergent and find its limit.

Solution. Since $0 < b_{n+1} = a^{n+1} < a^n = b_n < 1$ then (b_n) is bounded and monotonically decreasing.

So it is convergent, let $A = \lim_{n \rightarrow \infty} b_n$. Then

$$A = \lim_{n \rightarrow \infty} b_{n+1} = \lim_{n \rightarrow \infty} a \cdot b_n = a \cdot A \iff A(1 - a) = 0, \text{ so } A = 0.$$

Exercise 2. Let $a_1 = 4$ and $a_{n+1} = 8 - \frac{15}{a_n}$. Prove that the sequence (a_n) is convergent and find its limit.

Solution. The first few terms of the sequence:

$$a_1 = 4, a_2 = 4.25, a_3 = 4.47059, a_4 = 4.64474, a_5 = 4.77054, \dots$$

1) First we calculate the possible limits of (a_n) . If (a_n) is convergent then

$$A = \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} a_{n+1} = 8 - \frac{15}{A} \implies A^2 - 8A + 15 = (A - 3)(A - 5) = 0, \text{ therefore } A = 3 \text{ or } A = 5.$$

2) Next, we investigate the boundedness and monotonicity of (a_n) . If we prove that (a_n) is bounded and monotonically increasing or decreasing, then (a_n) is convergent and its limit is the supremum or the infimum of the sequence.

(i) First we prove boundedness by induction, that is, we prove that $3 < a_n < 5$ for all $n \in \mathbb{N}^+$.

I. The statement is true for $n = 1$: $3 < a_1 = 4 < 5$.

II. Assume that $3 < a_n < 5$. Then

$$3 < a_n < 5 \implies \frac{1}{5} < \frac{1}{a_n} < \frac{1}{3} \implies 3 < \frac{15}{a_n} < 5 \implies -3 > -\frac{15}{a_n} > -5 \implies 3 < 8 - \frac{15}{a_n} = a_{n+1} < 5.$$

(ii) Next we prove by induction that (a_n) is monotonically increasing, that is, $a_n < a_{n+1}$ for all $n \in \mathbb{N}^+$.

I. The statement is true for $n = 1$: $a_1 = 4 < a_2 = \frac{17}{4} = 4.25$

II. Assume that $a_n < a_{n+1}$. Then

$$a_n < a_{n+1} \implies \frac{1}{a_n} > \frac{1}{a_{n+1}} \text{ (since } a_n > 3 > 0) \implies -\frac{15}{a_n} < -\frac{15}{a_{n+1}} \implies a_{n+1} = 8 - \frac{15}{a_n} < 8 - \frac{15}{a_{n+1}} = a_{n+2}.$$

Since (a_n) is monotonic increasing and bounded then (a_n) is convergent. The limit of (a_n) cannot be $A = 3$, since $a_1 = 4$ and the sequence is monotonic increasing. Therefore $\lim_{n \rightarrow \infty} a_n = 5$.

The sequence $a_n = \left(1 + \frac{1}{n}\right)^n$

Theorem. The sequence $a_n = \left(1 + \frac{1}{n}\right)^n$ is monotonically increasing and bounded, so it is convergent.

1st proof. a) Monotonicity. We use the inequality between the arithmetic and geometric means:

$$\text{if } a_1, a_2, \dots, a_k \geq 0 \text{ then } \sqrt[k]{a_1 a_2 \dots a_k} \leq \frac{a_1 + a_2 + \dots + a_k}{k}.$$

$$\text{Let } a_1 = \dots = a_n = 1 + \frac{1}{n} \text{ and } a_{n+1} = 1. \text{ Then } \sqrt[n+1]{\left(1 + \frac{1}{n}\right)^n \cdot 1} \leq \frac{n\left(1 + \frac{1}{n}\right) + 1}{n+1} = 1 + \frac{1}{n+1},$$

$$\text{so } a_n = \left(1 + \frac{1}{n}\right)^n \leq \left(1 + \frac{1}{n+1}\right)^{n+1} = a_{n+1} \text{ for all } n \in \mathbb{N}.$$

b) Boundedness. We use the inequality between the arithmetic and geometric means

$$\text{for the numbers } a_1 = \dots = a_n = 1 + \frac{1}{n} \text{ and } a_{n+1} = a_{n+2} = \frac{1}{2}. \text{ Then}$$

$$\sqrt[n+2]{\left(1 + \frac{1}{n}\right)^n \cdot \frac{1}{4}} \leq \frac{n\left(1 + \frac{1}{n}\right) + 2 \cdot \frac{1}{2}}{n+2} = 1, \quad \text{so } a_n = \left(1 + \frac{1}{n}\right)^n \leq 4 \text{ for all } n \in \mathbb{N}.$$

2nd proof with the binomial theorem

a) **Boundedness.** $a_n = \left(1 + \frac{1}{n}\right)^n = \sum_{k=0}^n \binom{n}{k} \left(\frac{1}{n}\right)^k = 1 + 1 + \sum_{k=2}^n \frac{n(n-1) \dots (n-(k-1))}{k!} \cdot \frac{1}{n^k} =$

$$= 1 + 1 + \sum_{k=2}^n \frac{1}{k!} \cdot \frac{n}{n} \cdot \frac{n-1}{n} \cdot \dots \cdot \frac{n-(k-1)}{n} < 1 + 1 + \sum_{k=2}^n \frac{1}{k!} \cdot 1 \cdot \dots \cdot 1 = \sum_{k=0}^n \frac{1}{k!} := s_n.$$

The sequence (s_n) is bounded above since the terms can be estimated from above by the terms of a geometric sequence with ratio $\frac{1}{2}$:

$$s_n = 1 + 1 + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} + \dots + \frac{1}{1 \cdot 2 \cdot \dots \cdot n} < 1 + \left(1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots + \frac{1}{2^{n-1}}\right) \leq$$

$$\leq 1 + \frac{1 - \left(\frac{1}{2}\right)^n}{1 - \frac{1}{2}} = 3 - \left(\frac{1}{2}\right)^{n-1} < 3. \quad \text{So } a_n = \left(1 + \frac{1}{n}\right)^n < s_n = \sum_{k=0}^n \frac{1}{k!} < 3.$$

b) **Monotonicity.**

$$a_{n+1} = \left(1 + \frac{1}{n+1}\right)^{n+1} = \sum_{k=0}^{n+1} \binom{n+1}{k} \left(\frac{1}{n+1}\right)^k = 2 + \sum_{k=2}^{n+1} \frac{1}{k!} \cdot \frac{n+1}{n+1} \cdot \frac{n}{n+1} \cdot \frac{n-1}{n+1} \cdot \dots \cdot \frac{(n+1)-(k-1)}{n+1} =$$

$$= 2 + \sum_{k=2}^{n+1} \frac{1}{k!} \left(1 - \frac{1}{n+1}\right) \left(1 - \frac{2}{n+1}\right) \dots \left(1 - \frac{k-1}{n+1}\right) =$$

$$= 2 + \sum_{k=2}^n \frac{1}{k!} \left(1 - \frac{1}{n+1}\right) \dots \left(1 - \frac{k-1}{n+1}\right) + \frac{(n+1)}{(n+1)} \frac{1}{(n+1)^{n+1}} >$$

$$> 2 + \sum_{k=2}^n \frac{1}{k!} \left(1 - \frac{1}{n}\right) \dots \left(1 - \frac{k-1}{n}\right) + 0 = a_n. \quad \text{So } a_n < a_{n+1}.$$

Definition: The number e is defined as the limit of the above sequence:

$$e := \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n.$$

Remark: From the 2nd proof it follows that $2 < e < 3$.

Some terms of the sequence are: $a_1 = 2, a_2 = 2.25, a_3 \approx 2.37, a_4 \approx 2.44, a_5 \approx 2.488$

$$a_{10} \approx 2.59, a_{20} \approx 2.65, a_{100} \approx 2.70481, a_{200} \approx 2.71152$$

$$a_{1000} \approx 2.71692, a_{10000} \approx 2.71815$$

Theorems. 1) The number $e \approx 2.718281828459045235360287 \dots$ is irrational.

2) $\lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n = e^x$ for all $x \in \mathbb{R}$

3) If $x_n \xrightarrow{n \rightarrow \infty} \infty$, then $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{x_n}\right)^{x_n} = e$.

4) $e = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{n!}\right) = \lim_{n \rightarrow \infty} \sum_{k=0}^n \frac{1}{k!} = \sum_{k=0}^{\infty} \frac{1}{k!}$

Remark. The convergence of the series $\sum_{n=0}^{\infty} \frac{1}{n!}$ is very fast, for example

$$\sum_{n=0}^6 \frac{1}{n!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} \approx 2.718 \dots \text{ (3 digits are accurate)}$$

$$\sum_{n=0}^{10} \frac{1}{n!} \approx 2.7182818 \dots \text{ (7 digits are accurate)}$$

$$\sum_{n=0}^{15} \frac{1}{n!} \approx 2.71828182845 \dots \text{ (11 digits are accurate)}$$

$$\sum_{n=0}^{20} \frac{1}{n!} \approx 2.7182818284590452353 \dots \text{ (19 digits are accurate)}$$

Exercises

The sequence $a_n = \left(1 + \frac{1}{n}\right)^n$

$$1. a_n = \left(1 + \frac{1}{n^3 + n + 6}\right)^{n^3 + n + 6} \rightarrow e, \text{ since it is a subsequence of } \left(1 + \frac{1}{n}\right)^n.$$

$$2. a_n = \left(1 + \frac{1}{n-6}\right)^n = \left(1 + \frac{1}{n-6}\right)^{n-6} \cdot \left(1 + \frac{1}{n-6}\right)^6 \rightarrow e \cdot 1^6 = e$$

$$3. a_n = \left(1 + \frac{1}{6n+1}\right)^{6n-7} = \left(1 + \frac{1}{6n+1}\right)^{6n+1} \cdot \frac{1}{\left(1 + \frac{1}{6n+1}\right)^8} \rightarrow e \cdot \frac{1}{1^8} = e$$

$$4. a_n = \left(\frac{n+3}{n+4}\right)^{n-2} = \left(\frac{n+4-1}{n+4}\right)^{n+4-6} = \left(1 + \frac{-1}{n+4}\right)^{n+4} \cdot \frac{1}{\left(\frac{n+4}{n+3}\right)^6} \rightarrow e^{-1} \cdot \frac{1}{1^6} = \frac{1}{e}$$

Here we used that $\frac{n+4}{n+3} = \frac{1 + \frac{4}{n}}{1 + \frac{3}{n}} \rightarrow \frac{1+0}{1+0} = 1.$

Another solution: $a_n = \left(\frac{n+3}{n+4}\right)^{n-2} = \frac{\left(1 + \frac{3}{n}\right)^n}{\left(1 + \frac{4}{n}\right)^n} \cdot \left(\frac{n+4}{n+3}\right)^2 \rightarrow \frac{e^3}{e^4} \cdot 1^2 = \frac{1}{e}$

$$5. a_n = \left(\frac{n^2-2}{n^2+3}\right)^{n^2} = \frac{\left(1 + \frac{-2}{n^2}\right)^{n^2}}{\left(1 + \frac{3}{n^2}\right)^{n^2}} \rightarrow \frac{e^{-2}}{e^3} = e^{-5}$$

$$6. a_n = \left(\frac{n+1}{n+6}\right)^{2n} = \left(\frac{\left(1 + \frac{1}{n}\right)^n}{\left(1 + \frac{6}{n}\right)^n}\right)^2 \rightarrow \left(\frac{e^1}{e^6}\right)^2 = (e^{-5})^2 = e^{-10}$$

$$7. a_n = \left(\frac{2n+2}{2n+9}\right)^{2n} = \frac{\left(1 + \frac{2}{2n}\right)^{2n}}{\left(1 + \frac{9}{2n}\right)^{2n}} \rightarrow \frac{e^2}{e^9} = e^{-7}$$

$$8. \text{ Calculate the limit of } a_n = \left(\frac{2n^2+5}{2n^2+3}\right)^{4n^2}$$

$$\text{1st solution. } a_n = \left(\frac{\left(1 + \frac{5}{2n^2}\right)^{2n^2}}{\left(1 + \frac{3}{2n^2}\right)^{2n^2}}\right)^2 \rightarrow \left(\frac{e^5}{e^3}\right)^2 = e^4$$

$$\text{2nd solution. } a_n = \frac{\left(1 + \frac{5 \cdot 2}{4n^2}\right)^{4n^2}}{\left(1 + \frac{3 \cdot 2}{4n^2}\right)^{4n^2}} \rightarrow \frac{e^{10}}{e^6} = e^4$$

9. Calculate the limit of the following sequences:

$$a_n = \left(\frac{3n^2+1}{3n^2-2}\right)^{3n^2}, \quad b_n = \left(\frac{3n^2+1}{3n^2-2}\right)^{9n^2}$$

$$c_n = \left(\frac{3n^2+1}{3n^2-2}\right)^{3n^3}, \quad d_n = \left(\frac{3n^2+1}{3n^2-2}\right)^{3n}$$

$$\text{Solution. } a_n = \left(\frac{3n^2+1}{3n^2-2}\right)^{3n^2} = \frac{\left(1 + \frac{1}{3n^2}\right)^{3n^2}}{\left(1 + \frac{-2}{3n^2}\right)^{3n^2}} \rightarrow \frac{e}{e^{-2}} = e^3 = A$$

$$b_n = \left(\frac{3n^2+1}{3n^2-2}\right)^{9n^2} = (a_n)^3 \Rightarrow b_n \rightarrow A^3 = e^9$$

$$c_n = \left(\frac{3n^2+1}{3n^2-2}\right)^{3n^3} = (a_n)^n$$

In the estimation below we use that $2 < e < 3$, so $e^3 > 2^3 = 8$.

Since $a_n \rightarrow e^3$ then $\exists N_1$ such that if $n > N_1$ then $c_n = (a_n)^n > 8^n \rightarrow \infty \Rightarrow c_n \rightarrow \infty$

$$d_n = \left(\frac{3n^2+1}{3n^2-2}\right)^{3n} = \sqrt[n]{a_n}$$

Since $a_n \rightarrow e^3$ then for $\varepsilon = 0.1 \exists N_2$ such that if $n > N_2$ then $\sqrt[n]{e^3 - 0.1} \leq d_n \leq \sqrt[n]{e^3 + 0.1}$.

Since $\sqrt[n]{e^3 - 0.1} \rightarrow 1$ and $\sqrt[n]{e^3 + 0.1} \rightarrow 1$ then by the Sandwich Theorem $d_n \rightarrow 1$.

Recursive sequences

1. Let $a_1 = \frac{4}{3}$ and $a_{n+1} = \frac{3 + a_n^2}{4}$, $n = 1, 2, \dots$

Prove that the sequence (a_n) is convergent and find its limit.

Solution. $a_1 \approx 1.33 > a_2 = \frac{3 + \left(\frac{4}{3}\right)^2}{4} \approx 1.194 > a_3 \approx 1.1067$

Conjecture: (a_n) is monotonically decreasing, so $a_n > a_{n+1} > 0$.

Proof: by induction.

I. $a_1 > a_2 > a_3 > 0$ is satisfied.

II. Assume that $a_{n-1} > a_n$. From the definition of the sequence it is obvious that $a_n > 0$

$\left(a_n = \frac{3 + a_{n-1}^2}{4} \geq \frac{3}{4} > 0\right)$. Then

$$\begin{aligned} a_{n-1} > a_n > 0 &\implies a_{n-1}^2 > a_n^2 \implies 3 + a_{n-1}^2 > 3 + a_n^2 \implies a_n = \frac{3 + a_{n-1}^2}{4} > \frac{3 + a_n^2}{4} = a_{n+1} \\ &\implies a_n > a_{n+1}. \end{aligned}$$

Since (a_n) is monotonic decreasing and bounded below (since $a_n > 0$) then (a_n) is

convergent, therefore $A = \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{3 + a_n^2}{4}$

$$\implies A = \frac{3 + A^2}{4} \implies A^2 - 4A + 3 = (A - 1)(A - 3) = 0 \implies A = 1 \text{ or } A = 3.$$

Since $a_n < a_1 = \frac{4}{3}$ then $A = 3$ cannot be the case, so $A = \lim_{n \rightarrow \infty} a_n = 1$.

2. Let $a_1 = 1$ and $a_{n+1} = \sqrt{6 + a_n}$, $n = 1, 2, \dots$

Is the sequence convergent? If so, what is the limit?

Solution. The first few terms of the sequence: $a_1 = 1$, $a_2 \approx 2.646$, $a_3 \approx 2.94$, ...

Since $\sqrt{6 + a_n} \geq 0$ then the terms of the sequence are positive.

1) First we calculate the possible limits of (a_n) . If (a_n) is convergent then

$$A = \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} a_{n+1} = \lim_{n \rightarrow \infty} \sqrt{6 + a_n} = \sqrt{6 + A} \implies A^2 - A - 6 = (A - 3)(A + 2) = 0, \text{ from where}$$

$A = 3$ or $A = -2$. Since $a_n = \sqrt{6 + a_{n-1}} > 0$ then $A = -2$ cannot be the case, so the only possible limit is $A = 3$.

2) Next, we investigate the boundedness and monotonicity of (a_n) .

(i) First we prove by induction that (a_n) is monotonically increasing and the terms are positive, that is, $0 < a_n < a_{n+1}$ for all $n \in \mathbb{N}^+$.

I. The statement is true for $n = 1$: $0 < a_1 = 1 < a_2 = \sqrt{7} \approx 2.646$

II. Assume that $0 < a_n < a_{n+1}$. Then

$$0 < a_n < a_{n+1} \implies 0 < 0 + 6 < 6 + a_n < 6 + a_{n+1} \implies 0 < \sqrt{6 + a_n} < \sqrt{6 + a_{n+1}} \implies 0 < a_{n+1} < a_{n+2}.$$

(ii) Next we prove that the sequence is bounded above. $A = 3$ is a suitable choice for the upper bound, that is, we show that $a_n < 3$ for all $n \in \mathbb{N}^+$.

I. The statement is true for $n = 1$: $a_1 = 1 < 3$

II. Assume that $a_n < 3$. Then $a_{n+1} = \sqrt{6 + a_n} < \sqrt{6 + 3} = 3$.

Since (a_n) is monotonic increasing and bounded above then (a_n) is convergent, so $\lim_{n \rightarrow \infty} a_n = 3$.

We have seen that this is the only possible limit.

Remark. Monotonicity can also be proved as follows.

$$0 < a_n < a_{n+1} = \sqrt{6 + a_n} \iff a_n^2 < 6 + a_n \iff a_n^2 - a_n - 6 < 0 \iff -2 < a_n < 3.$$

Here $-2 < a_n$ trivially holds, since $a_n > 0$, and $a_n < 3$ can be proved by induction.

3. Let $a_1 = -3$ and $a_{n+1} = \frac{5 - 6a_n^2}{13}$, $n = 1, 2, \dots$. Is the sequence convergent?

Solution. $a_1 = -3$, $a_2 \approx -3.769$, $a_3 \approx -6.1725$, ...

Is the sequence monotonic decreasing?

$$a_{n+1} = \frac{5 - 6a_n^2}{13} < a_n \iff 6a_n^2 + 13a_n - 5 > 0 \quad \left(6x^2 + 13x - 5 = 0 \iff x_1 = -\frac{5}{2}, x_2 = \frac{1}{3} \right)$$

It means that the sequence is monotonic decreasing if and only if $a_n < -\frac{5}{2}$ or $a_n > \frac{1}{3}$.

Homework: It can be proved by induction that $a_n \leq -3 \left(< -\frac{5}{2} \right)$.

Therefore the sequence is monotonic decreasing with initial value $a_1 = -3$.

If the sequence were bounded from below then it would be convergent and for the limit

$$\text{we would have } A = \frac{5 - 6A^2}{13} \implies \text{the possible values of } A \text{ could be } A = -\frac{5}{2} \text{ or } A = \frac{1}{3}.$$

Since $a_n \leq -3$ for all n then these numbers cannot be the limit, so (a_n) is not convergent and therefore not bounded from below. Since (a_n) is monotonic decreasing then

$$\lim_{n \rightarrow \infty} a_n = -\infty.$$