The Ratio Test and the Root Test, Exercises

Exercise 1: Decide whether the following series are convergent or divergent.

a) $\sum_{n=1}^{\infty} \frac{9^{n-2}}{n!}$ b) $\sum_{n=1}^{\infty} \frac{5^{3n}}{n^4}$ c) $\sum_{n=1}^{\infty} \frac{(n+1)!}{n^n}$

Solution:

a) Let $a_n := \frac{9^{n-2}}{n!}$ and let us apply the Ratio Test: $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{9^{(n+1)-2}\ n!}{(n+1)!\ 9^{n-2}} =$

$$\lim_{n\to\infty} \frac{9}{n+1} = 0 < 1 \quad \Longrightarrow \quad \sum_{n=0}^{\infty} a_n \text{ is convergent}$$

b) Let $a_n := \frac{5^{3n}}{n^4}$. The Ratio Test can be applied but the Root Test is more convenient:

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \frac{5^3}{\sqrt[n]{n^4}} = 5^3 \lim_{n\to\infty} \frac{1}{\left(\sqrt[n]{n}\right)^4} = 5^3 > 1 \quad \Longrightarrow \quad \sum_{n=0}^{\infty} a_n \text{ is divergent}$$

c) Let $a_n := \frac{(n+1)!}{n^n}$. Here we apply the Ratio Test:

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+2)!\ n^n}{(n+1)^{n+1}\ (n+1)!} = \lim_{n\to\infty} \frac{(n+2)\ n^n}{(n+1)^{n+1}} =$$

$$= \lim_{n\to\infty} \frac{n+2}{n+1} \left(\frac{n}{n+1}\right)^n = = \lim_{n\to\infty} \frac{1+\frac{2}{n}}{1+\frac{1}{n}} \frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{\mathrm{e}} < 1 \quad \Longrightarrow \quad \sum_{n=0}^{\infty} a_n \text{ is convergent}$$

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Exercise 2: Is the following series convergent?

$$\sum_{n=1}^{\infty} \frac{(n+5)\ 3^{n-1}}{5^{n+1}}$$

Solution:

Let $a_n := \frac{(n+5)\ 3^{n-1}}{5^{n+1}}$. If we apply the Root Test, then the convergence of the sequence $\sqrt[n]{n+5}$ should be proved by the Sandwich Theorem, so it is more convenient to use the Ratio Test.

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+6)\ 3^n\ 5^{n+1}}{5^{n+2}\ (n+5)\ 3^{n-1}} = \lim_{n\to\infty} \frac{3}{5} \frac{n+6}{n+5} =$$

$$= \lim_{n\to\infty} \frac{3}{5} \frac{1+\frac{6}{n}}{1+\frac{5}{n}} = \frac{3}{5} < 1 \quad \Longrightarrow \quad \sum_{n=0}^{\infty} a_n \text{ is convergent}$$

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Exercise 3: Is the following series convergent?

$$\sum_{n=1}^{\infty} \frac{n^4\ (3n+3)^{n^2}}{(3n+1)^{n^2}}$$

Solution:

Let $a_n := \dfrac{n^4 \; (3n+3)^{n^2}}{(3n+1)^{n^2}}$. By applying the Root Test, we get that

$$\lim_{n\to\infty} \sqrt[n]{a_n} \;=\; \lim_{n\to\infty} \sqrt[n]{n^4}\,\left(\frac{3n+3}{3n+1}\right)^n \;=\; \lim_{n\to\infty} \left(\sqrt[n]{n}\right)^4 \; \frac{\left(1+\dfrac{3/3}{n}\right)^n}{\left(1+\dfrac{1/3}{n}\right)^n} \;=$$

$$= 1^4 \; \frac{\mathrm{e}}{\mathrm{e}^{1/3}} \;=\; \mathrm{e}^{2/3} \;>\; 1 \qquad \Longrightarrow \quad \sum_{n=0}^{\infty} a_n \;\text{ is divergent}$$

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Exercise 4: Is the following series convergent?

$$\sum_{n=1}^{\infty} \left(\frac{3+n^2}{2+n^2}\right)^{n^3} \; \frac{n^5}{2^{2n+1}}$$

Solution:

Let $a_n := \left(\dfrac{3+n^2}{2+n^2}\right)^{n^3} \; \dfrac{n^5}{2^{2n+1}}$. By applying the Root Test, we get that

$$\lim_{n\to\infty} \sqrt[n]{a_n} \;=\; \ldots \;=\; \lim_{n\to\infty} \frac{\left(1+\dfrac{3}{n^2}\right)^{n^2}}{\left(1+\dfrac{2}{n^2}\right)^{n^2}} \; \frac{\left(\sqrt[n]{n}\right)^5}{4\cdot\sqrt[n]{2}} \;=\; \frac{\mathrm{e}^3}{\mathrm{e}^2}\;\frac{1^5}{4\cdot 1} \;=\; \frac{\mathrm{e}}{4} \;<\; 1 \quad \Longrightarrow \quad \sum_{n=0}^{\infty} a_n \;\text{ is convergent}$$

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Exercise 5: Decide whether the following series are convergent or divergent.

a) $\displaystyle\sum_{n=0}^{\infty} \left(\frac{n^2-2}{n^2+5}\right)^{n^2}$ b) $\displaystyle\sum_{n=0}^{\infty} \left(\frac{n^2-2}{n^2+5}\right)^{n}$ c) $\displaystyle\sum_{n=0}^{\infty} \left(\frac{n^2-2}{n^2+5}\right)^{n^3}$

Solution:

a) Let $a_n := \left(\dfrac{n^2-2}{n^2+5}\right)^{n^2}$. Then $\displaystyle\lim_{n\to\infty} a_n \;=\; \lim_{n\to\infty} \frac{\left(1+\dfrac{-2}{n^2}\right)^{n^2}}{\left(1+\dfrac{5}{n^2}\right)^{n^2}} \;=\; \frac{\mathrm{e}^{-2}}{\mathrm{e}^{5}} \;=\; \mathrm{e}^{-7} \neq 0$

Since the general term doesn't converge to 0, then the series $\displaystyle\sum_{n=0}^{\infty} a_n$ is divergent by the nth Term Test.

b) Let $b_n := \displaystyle\sum_{n=0}^{\infty} \left(\frac{n^2-2}{n^2+5}\right)^{n} \;=\; \sqrt[n]{a_n}$.

$$\lim_{n\to\infty} a_n \;=\; \mathrm{e}^{-7} \quad \Longrightarrow \quad \mathrm{e}^{-7} - \frac{\mathrm{e}^{-7}}{2} \;<\; a_n \;<\; \mathrm{e}^{-7} + \frac{\mathrm{e}^{-7}}{2}\;, \;\text{ if } n > N_0$$

$$\Longrightarrow \quad \underbrace{\sqrt[n]{\frac{1}{2}\,\mathrm{e}^{-7}}}_{\to\,1} \;<\; \sqrt[n]{a_n} \;<\; \underbrace{\sqrt[n]{\frac{3}{2}\,\mathrm{e}^{-7}}}_{\to\,1} \quad \Longrightarrow \quad b_n \;=\; \sqrt[n]{a_n} \to\; 1\,.$$

Since $\displaystyle\lim_{n\to\infty} b_n = 1 \neq 0$, then the series $\displaystyle\sum_{n=0}^{\infty} b_n$ as also divergent by the nth Term Test.

c) Let $c_n := \sum_{n=0}^{\infty} \left(\frac{n^2-2}{n^2+5}\right)^{n^3} = {a_n}^n$. By applying the Root Test, we get that

$$\lim_{n\to\infty} \sqrt[n]{c_n} = \lim_{n\to\infty} a_n = e^{-7} < 1 \quad \Longrightarrow \quad \sum_{n=0}^{\infty} c_n \text{ is convergent}$$

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Exercise 6: Is the following series convergent?

$$\sum_{n=0}^{\infty} \frac{2^n + 3^{n+2} + (\frac{1}{2})^n}{(2n)! + 3n^2}$$

Solution: $$c_n := \frac{2^n + 3^{n+2} + (\frac{1}{2})^n}{(2n)! + 3n^2} < \frac{3^n + 9\cdot 3^n + 3^n}{(2n)!} = 11\,\frac{3^n}{(2n)!} := d_n$$

Using the Ratio Test, it can be proved that $\sum_{n=0}^{\infty} d_n$ is convergent (homework). Therefore, the series $\sum_{n=0}^{\infty} c_n$ is also convergent by the Comparison Test.

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Exercise 7: Prove that the following series are convergent. Estimate the error if the sum of the series is approximated by the sum of the first 100 terms.

a) $\sum_{n=0}^{\infty} \frac{(n+2)\,3^{n-1}}{(n+5)\,n!}$ b) $\sum_{n=1}^{\infty} \left(\frac{n+2}{6n-1}\right)^{3n}$

Solution:

a) Let $a_n := \frac{(n+2)\,3^{n-1}}{(n+5)\,n!}$, then $a_n < \frac{3^{n-1}}{n!} := b_n \quad \sum_{n=0}^{\infty} b_n$.

The convergence of $\sum_{n=0}^{\infty} b_n$ can be shown using the Ratio Test:

$$\lim_{n\to\infty} \frac{b_{n+1}}{b_n} = \lim_{n\to\infty} \frac{3^n\, n!}{(n+1)!\,3^{n-1}} = \lim_{n\to\infty} \frac{3}{n+1} = 0 < 1$$

$$\Longrightarrow \sum_{n=0}^{\infty} b_n \text{ is convergent} \underbrace{\Longrightarrow}_{\text{Comparison Test}} \sum_{n=0}^{\infty} a_n \text{ is convergent}$$

The error for the approximation $s \approx_{100}$ is:

$$0 < E = \sum_{n=0}^{\infty} a_n - \sum_{n=0}^{100} a_n = \sum_{n=101}^{\infty} \frac{(n+2)\,3^{n-1}}{(n+5)\,n!} < \sum_{n=101}^{\infty} \frac{3^{n-1}}{n!} = \frac{3^{100}}{101!} + \frac{3^{101}}{102!} + \frac{3^{102}}{103!} + \cdots =$$

$$= \frac{3^{100}}{101!}\left(1 + \frac{3}{102} + \frac{3^2}{102\cdot 103} + \cdots\right) < \frac{3^{100}}{101!}\,(1 + \frac{3}{102} + \frac{3^2}{102^2} + \cdots) =$$

$$= \frac{3^{100}}{101!}\,\frac{1}{1-\frac{3}{102}} \quad \left(\text{geometric series with } r = \frac{3}{102}\right)$$

b) Let $a_n := \left(\dfrac{n+2}{6n-1}\right)^{3n}$. Then

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \left(\frac{n+2}{6n-1}\right)^3 = \lim_{n\to\infty} \left(\frac{1+\frac{2}{n}}{6-\frac{1}{n}}\right)^3 = \frac{1}{6^3} < 1 \qquad \Longrightarrow \qquad \sum_{n=0}^{\infty} a_n \text{ is convergent}$$

The error for the approximation $s \approx_{100}$ is:

$$0 < E = \sum_{n=101}^{\infty} \left(\frac{n+2}{6n-1}\right)^{3n} < \sum_{n=101}^{\infty} \left(\frac{n+2n}{6n-n}\right)^{3n} = \sum_{n=101}^{\infty} \left(\left(\frac{3}{5}\right)^3\right)^n =$$

$$= \left(\frac{3}{5}\right)^{303} \frac{1}{1-\left(\frac{3}{5}\right)^3} \qquad \left(\text{geometric series with } r = \left(\frac{3}{5}\right)^3\right)$$

Practice exercises

Exercise 8: Decide whether the following series are convergent or divergent.

a) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2n+3}{2n+1}\right)^{n^2+3n}$ b) $\displaystyle\sum_{n=1}^{\infty} \frac{n!\ 6^{n-1}}{(2n)!}$ c) $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{\binom{2n}{n}}$

d) $\displaystyle\sum_{n=1}^{\infty} \frac{4^n\ (n+3)}{(n)!}$ e) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{(n+1)^{n+2}}$ f) $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^2}{3^n\ (2n)!}$

Exercise 9: Prove that the following series is convergent. Estimate the error if the sum of the series is approximated by the sum of the first 200 terms.

$$\sum_{n=1}^{\infty} \frac{2^{3n+1}}{(n)!}$$

Exercise 10: Prove that the following series is convergent. Estimate the error if the sum of the series is approximated by the sum of the first 100 terms.

$$\sum_{n=1}^{\infty} \frac{n}{(n+3)\ 6^{n+1}}$$