
Calculus 1 - Homework 3

1. (4 points) Let $A = \left\{-\frac{1}{n} : n \in \mathbb{N}\right\} \cup (\mathbb{Q} \cap [1, 2]) \cup (3, 4]$.

Find the set of interior points, boundary points, limit points and isolated points of A .

2. (3+3 points) Calculate the following limits:

a) $\lim_{x \rightarrow 1} \frac{x^2 - 1}{\sqrt{x} - \sqrt{2-x}}$ **b)** $\lim_{x \rightarrow 0} \frac{\sin^2(ax)}{\cos(bx) - 1}$, where $a, b \in \mathbb{R} \setminus \{0\}$.

3. (4 points) Choose the values of the parameters $a, b \in \mathbb{R}$ so that the following function be continuous on \mathbb{R} :

$$f(x) = \begin{cases} \frac{\cos^2 x - a}{x} & \text{if } x < 0 \\ \sin^2 \frac{\pi(x+b)}{2} & \text{if } x \geq 0 \end{cases}$$

4. (3 points) Are the following statements true or false? Give a reason for your answer.

- a) There exists a continuous function $f : (-1, 1) \rightarrow \mathbb{R}$ whose range is $[0, 1]$.
- b) There exists a continuous function $f : [-1, 1] \rightarrow \mathbb{R}$ whose range is $(0, 1)$.
- c) There exists a continuous function $f : [-1, 1] \rightarrow \mathbb{R}$ whose range is $[1, 2] \cup [4, 5]$.

5. (5 points) Determine the points of discontinuities of the following functions. What type of discontinuities are these?

a) $f(x) = e^{-\frac{1}{x^2}}$ b) $g(x) = \frac{1}{1 - e^x}$ c) $h(x) = \frac{1}{1 - e^{\frac{1}{x}}}$

6. (3 points) Let $f(x) = e^{-x} \cos(\pi x) + x^3 - 4$. Prove that f has a zero in the open interval $(0, 2)$.

7.* (4 points) Prove that if f is continuous on $[a, \infty)$ and $\exists \lim_{x \rightarrow \infty} f(x) = A \in \mathbb{R}$ then f is uniformly continuous on $[a, \infty)$.

Deadline: November 17th