Calculus 1 - Homework 1.

1. (3 points) Decide whether the statement is true or false and write down the negation of the statement: $\forall r > 0 (\forall x \in \mathbb{R} (\exists q \in \mathbb{Q} (|x - q| < r)))$.

2. (4 points) Let $a_0 = 5$ and $a_{n+1} = 8 - \frac{12}{a_n}$. Prove that $\forall n \in \mathbb{N}\ (2 \le a_n \le 6)$.

3. (4 points) What is the maximum value of xy if $x, y \ge 0$ and $2x + 3y = 10$?

4. (4 points) Given a right angled triangle, its sides are a, b and c where c is the hypotenuse. Prove that $a + b \le \sqrt{2} \cdot c$. When does equality hold?

5. (4 points) Let $a_n = \frac{6n^4 - n^3 + 100}{2n^4 + n - 1000}$. Find the limit of a_n and provide a threshold index N for $\varepsilon = 0.01$.

6. (3 points) Prove that if $\lim_{n\to\infty} a_n = \infty$, then $\lim_{n\to\infty} \sqrt[k]{a_n} = \infty$ for all $k \in \mathbb{N}$.

7. (3 points) Find the limit of the sequence $a_n = \sqrt{n^2 + n - 2} - \sqrt{n^2 - 2n + 3}$.

Deadline: September 27th

Solutions

1. (3 points) Decide whether the statement is true or false and write down the negation of the statement: $\forall r > 0 (\forall x \in \mathbb{R} (\exists q \in \mathbb{Q} (|x - q| < r)))$.

Solution. True. Negation: $\exists r > 0 (\exists x \in \mathbb{R} (\forall q \in \mathbb{Q} (|x - q| \ge r)))$

2. (4 points) Let $a_0 = 5$ and $a_{n+1} = 8 - \frac{12}{a_n}$. Prove that $\forall n \in \mathbb{N}\ (2 \le a_n \le 6)$.

Solution: By the method of induction:
I. The statement is true for $n = 0$: $2 \le a_0 = 3 \le 6$
II. Assume that $2 < a_n < 6$. Then
$$2 \le a_n \le 6 \Longrightarrow \frac{1}{6} \le \frac{1}{a_n} \le \frac{1}{2} \Longrightarrow 2 \le \frac{12}{a_n} \le 6 \Longrightarrow -2 \ge -\frac{12}{a_n} \ge -6$$
$$\Longrightarrow 2 \le 8 - \frac{12}{a_n} = a_{n+1} \le 6.$$

3. (4 points) What is the maximum value of xy if $x, y \ge 0$ and $2x + 3y = 10$?

Solution: We apply the inequality of arithmetic and geometric means for $a = 2x$ and $b = 3y$. Then $\sqrt{2x\cdot 3y} \leq \frac{2x+3y}{2} = \frac{10}{2} = 5 \Longrightarrow 6xy \leq 25 \Longrightarrow xy \leq \frac{25}{6}$. Thus the maximum of xy is $\frac{25}{6}$ and equality holds if and only if $2x = 3y$ and $2x + 3y = 10$ from where $x = \frac{5}{2}$ and $y = \frac{5}{3}$.

4. (4 points) Given a right angled triangle, its sides are a, b and c where c is the hypotenuse. Prove that $a + b \leq \sqrt{2}\cdot c$. When does equality hold?

Solution: By the Pythagorean theorem $c = a^2 + b^2$. Apply the inequality of arithmetic and quadratic means for a and b, then $\frac{a+b}{2} \leq \sqrt{\frac{a^2+b^2}{2}} = \sqrt{\frac{c^2}{2}} \Longrightarrow a + b \leq \sqrt{2}\cdot c$.

Equality holds if and only if $a = b$, that is, for isosceles triangles.

5. (4 points) Let $a_n = \frac{6n^4 - n^3 + 100}{2n^4 + n - 1000}$. Find the limit of a_n and provide a threshold index N for $\varepsilon = 0.01$.

Solution. $a_n = \frac{6n^4 - n^3 + 100}{2n^4 + n - 1000} = \frac{n^4}{n^4}\,\frac{6 - \frac{1}{n} + \frac{100}{n^4}}{2 - \frac{1}{n^3} - \frac{1000}{n^4}} \longrightarrow \frac{6-0+0}{2-0-0} = 3.$

Let $\varepsilon > 0$ be fixed. By the definition we have to provide a threshold index $N(\varepsilon) \in \mathbb{N}$ such that if $n > N(\varepsilon)$ then $|\, a_n - A \,| < \varepsilon$.

$$|\, a_n - A \,| = \left| \frac{6n^4 - n^3 + 100}{2n^4 + n - 1000} - 3 \right| = \left| \frac{6n^4 - n^3 + 100 - 3\left(2n^4 + n - 1000\right)}{2n^4 + n - 1000} \right| =$$

$$= \left| \frac{-n^3 - 3n + 3100}{2n^4 + n - 1000} \right| = \left| \frac{n^3 + 3n - 3100}{2n^4 + n - 1000} \right|$$

It can be seen that if n is large enough, then both the numerator and the denominator are positive, so we can leave out the absolute value.

If $n \geq N_1$, then $n^3 + 3n - 3100 > 0$ (for example, $N_1 = 20$ or $N_1 = 100$ etc.)
If $n \geq N_2$, then $2n^4 + n - 1000 > 0$ (for example, $N_2 = 10$ or $N_2 = 100$ etc.)
Therefore,

$$\left| \frac{n^3 + 3n - 3100}{2n^4 + n - 1000} \right| \overset{n>N_1, n>N_2}{=} \frac{n^3 + 3n - 3100}{2n^4 + n \mathbf{-1000}} < \frac{n^3 + 3n^3 + 0}{2n^4 + 0 \mathbf{-n^4}} = \frac{4n^3}{n^4} = \frac{4}{n} < \varepsilon \Longleftrightarrow n > \frac{4}{\varepsilon}.$$

In the estimation we increase the terms in the numerator and decrease the terms in the denominator.

We used that $n^4 > 1000$ if $n > \sqrt[4]{1000} \approx 5.62$

So with the choice $N(\varepsilon) \geq \max\left\{N_1, N_2, 6, \left[\frac{4}{\varepsilon}\right]\right\}$ the definition holds.

If $\varepsilon = 0.01$ then $N(\varepsilon) \geq 400$.

6. (3 points) Prove that if $\lim_{n\to\infty} a_n = \infty$, then $\lim_{n\to\infty} \sqrt[k]{a_n} = \infty$ for all $k \in \mathbb{N}$.

Solution. Let $P > 0$ be arbitrary. Since $\lim_{n\to\infty} a_n = \infty$ then for P^k there exists $N \in \mathbb{N}$ such that $a_n > P^k$ if $n > N$. Then $\sqrt[k]{a_n} > P$ if $n > N$ also holds, so by the definition $\lim_{n\to\infty} \sqrt[k]{a_n} = \infty$.

7. (3 points) Find the limit of the sequence $a_n = \sqrt{n^2 + n - 2} - \sqrt{n^2 - 2n + 3}$.

Solution. $a_n = (\alpha - \beta) \cdot \frac{\alpha + \beta}{\alpha + \beta} = \frac{\alpha^2 - \beta^2}{\alpha + \beta} = \frac{(n^2 + n - 2) - (n^2 - 2n + 3)}{\sqrt{n^2 + n - 2} + \sqrt{n^2 - 2n + 3}} =$

$$= \frac{3n - 5}{\sqrt{n^2\left(1 + \frac{1}{n} - \frac{2}{n^2}\right)} + \sqrt{n^2\left(1 - \frac{2}{n} + \frac{3}{n^2}\right)}} = \frac{n}{n} \cdot \frac{3 - \frac{5}{n}}{\sqrt{1 + \frac{1}{n} - \frac{2}{n^2}} + \sqrt{1 - \frac{2}{n} + \frac{3}{n^2}}} \xrightarrow{n\to\infty}$$

$$\xrightarrow{n\to\infty} \frac{3 - 0}{\sqrt{1 + 0 - 0} + \sqrt{1 - 0 + 0}} = \frac{3}{2}$$