Calculus 1, Final exam 2, Part 2

16th January, 2023

Name: ____________________________________ **Neptun code:** _______________

1.: _____ **2.:** _____ **3.:** _____ **4.:** _____ **5.:** _____ **6.:** _____ **7.:** _____ **8.:** _____ **Sum:** _____

1. (10 points) Calculate the following limit: $\lim_{x\to 0} \frac{\sqrt{1+8x} - e^{4x}}{x \sin 2x}$

2. (5+5+5 points) Calculate the derivatives of the following functions:

a) $f : (0, \infty) \longrightarrow \mathbb{R}$, $f(x) = \left(x^2+1\right)^{\sin x}$ b) $f : (0, \infty) \longrightarrow \mathbb{R}$, $f(x) = \ln\left(\frac{\arctan(3x)}{\cos\left(x^2\right)+4}\right)$

c) $f : [0, \infty) \longrightarrow \mathbb{R}$, $f(x) = \sqrt{x}\, \sin\left(\sqrt{x}\right)$

In case c), use the definition to calculate $f_+'(0)$.

3. (10 points) We want to make a box with an open top from a 1 m by 1 m piece of cardboard, such that we cut out congruent squares from its corners, and turn up the sides. What are the sides of the box with maximal volume? What is this maximal volume?

4. (15 points) Analyze the following function and sketch its graph: $f(x) = (x-1)^2 e^{2x}$.

5. (10+10 points) Calculate the following integrals:

a) $I_1 = \int_0^{\ln 2} (2x+3)\, e^{-x}\, dx$ b) $I_2 = \int \frac{1}{x\sqrt{x} + 4\sqrt{x}}\, dx$ (substitution: $t = \sqrt{x}$)

6. (10+10 points) Calculate the following integrals:

a) $I_3 = \int \frac{x+2}{(x-1)\left(x^2+2\right)}\, dx$ b) $I_4 = \int \frac{e^{2x}}{e^{2x} - 5e^x + 6}\, dx$ (substitution: $t = e^x$)

7. (10 points) Consider the function $f(x) = \sqrt{\sin^3 x}$ on the interval $x \in [0, \pi]$. Rotate it around the x-axis and find the volume of the arising body.

8.* (10 points - BONUS)

The set of limit points of the set $A \subset \mathbb{R}$ is denoted by A'. Decide whether the following statements are true or false. In each case give a reason for your answer.

a) $A \subset A'$ for all $A \subset \mathbb{R}$.

b) $A' \subset A$ for all $A \subset \mathbb{R}$.

c) There exists $A \subset \mathbb{R}$ such that $A \neq \emptyset$ and $A' = \emptyset$.

d) There exists a bounded set $A \subset \mathbb{R}$ such that A has infinitely many elements and $A' = \emptyset$.

Solutions

1. (10 points) Calculate the following limit: $\lim\limits_{x\to 0}\dfrac{\sqrt{1+8x}-e^{4x}}{x\sin 2x}$

Solution. The limit has the form $\frac{0}{0}$, so the L'Hospital's rule can be applied:

$$\lim_{x\to 0}\frac{\sqrt{1+8x}-e^{4x}}{x\sin 2x}\overset{\frac{0}{0},L'H}{=}\lim_{x\to 0}\frac{\frac{1}{2}(1+8x)^{-\frac{1}{2}}\cdot 8-4e^{4x}}{\sin 2x+2x\cdot\cos 2x}\ \textbf{(4p)}\ \overset{\frac{0}{0},L'H}{=}\lim_{x\to 0}\frac{-\frac{1}{4}(1+8x)^{-\frac{3}{2}}\cdot 64-16e^{4x}}{2\cos 2x+2\cos 2x-4x\sin 2x}\ \textbf{(4p)}$$

$$=\frac{-\frac{1}{4}\cdot 64-16}{2+2-0}=-8\ \textbf{(2p)}$$

2. (5+5+5 points) Calculate the derivatives of the following functions:

a) $f:(0,\infty)\longrightarrow\mathbb{R},\ f(x)=(x^2+1)^{\sin x}$ b) $f:(0,\infty)\longrightarrow\mathbb{R},\ f(x)=\ln\left(\dfrac{\arctan(3x)}{\cos(x^2)+4}\right)$

c) $f:[0,\infty)\longrightarrow\mathbb{R},\ f(x)=\sqrt{x}\sin\left(\sqrt{x}\right)$

In case c), use the definition to calculate $f_+'(0)$.

Solution.

a) $f(x)=(x^2+1)^{\sin x}=e^{\ln\left((x^2+1)^{\sin x}\right)}=e^{(\sin x)\ln(x^2+1)}$

$$\Longrightarrow f'(x)=e^{(\sin x)\ln(x^2+1)}\cdot\left((\sin x)\cdot\ln(x^2+1)\right)'=(x^2+1)^{\sin x}\cdot\left(\cos x\cdot\ln(x^2+1)+\sin x\cdot\frac{2x}{x^2+1}\right)\ \textbf{(5p)}$$

b) $$f(x)=\ln\left(\frac{\arctan(3x)}{\cos(x^2)+4}\right)\Longrightarrow f'(x)=\frac{\cos(x^2)+4}{\arctan(3x)}\cdot\frac{\frac{1}{1+(3x)^2}\cdot 3\cdot(\cos(x^2)+4)-\arctan(3x)\cdot(-\sin(x^2))\cdot 2x}{(\cos(x)^2+4)^2}\ \textbf{(5p)}$$

c) If $x\neq 0$ then $f'(x)=\dfrac{1}{2\sqrt{x}}\sin\left(\sqrt{x}\right)+\sqrt{x}\cdot\cos\left(\sqrt{x}\right)\cdot\dfrac{1}{2\sqrt{x}}$ **(2p)**

If $x=0$ then by the definition of the derivative

$$f_+'(0)=\lim_{x\to 0+}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0+}\frac{\sqrt{x}\sin\left(\sqrt{x}\right)-0}{x-0}=\lim_{x\to 0+}\frac{\sin\left(\sqrt{x}\right)}{\sqrt{x}}=1\ \textbf{(3p)}$$

3. (10 points) We want to make a box with an open top from a 1 m by 1 m piece of cardboard, such that we cut out congruent squares from its corners, and turn up the sides. What are the sides of the box with maximal volume? What is this maximal volume?

Solution. Let x denote the sides of the squares. Then the base of the box is a square with sides $1-2x$ and the height of the box is x. The volume of the box is $V(x)=x(1-2x)^2=4x^3-4x^2+x$. We want to find the maximum of this function if $0<x<\frac{1}{2}$. **(3p)**

$V'(x)=12x^2-8x+1=0\iff x_1=\frac{1}{6},x_2=\frac{1}{2}$ **(3p)**

Because of the conditions, $x=\frac{1}{2}$ cannot be the case.

$V''(x) = 24x - 8$ and $V''\left(\frac{1}{6}\right) = 24 \cdot \frac{1}{6} - 8 = -4 < 0$, so V has a maximum at $x = \frac{1}{6}$. **(2p)**

The sides of the box with maximal volume are $\frac{2}{3}$, $\frac{2}{3}$ and $\frac{1}{6}$ m and the maximum of the volume is $\frac{2}{27}$ m^3. **(2p)**

4. (15 points) Analyze the following function and sketch its graph: $f(x) = (x-1)^2 e^{2x}$.

Solution.

1) The domain of f is $D_f = \mathbb{R} \setminus \{1\}$.

The zeros of f are: $(x-1)^2 e^{2x} \implies x = 1$

The limits of f at $\pm\infty$ are:

$$\lim_{x\to\infty} f(x) = \infty \cdot \infty = \infty, \quad \lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{(x-1)^2}{e^{-2x}} \overset{\frac{\infty}{\infty}, L'H}{=} \lim_{x\to-\infty} \frac{2(x-1)}{-2e^{-2x}} \overset{\frac{\infty}{\infty}, L'H}{=} \lim_{x\to-\infty} \frac{2}{4e^{-2x}} = \frac{2}{\infty} = 0. \ \textbf{(2p)}$$

2) $f'(x) = 2(x-1)e^{2x} + (x-1)^2 \cdot 2e^{2x} = 2e^{2x}(x-1)(1+x-1) = 2e^{2x}\left(x^2 - x\right) = 2e^{2x}(x-1)x = 0$

$\implies x_1 = 0, x_2 = 1$ **(2p)**

x	x<0	x=0	0<x<1	x=1	x>1
f'	+	0	−	0	+
f	↗	loc. max.	↘	loc. min.	↗

(3p)

$(f(0) = 1, \quad f(1) = 0)$

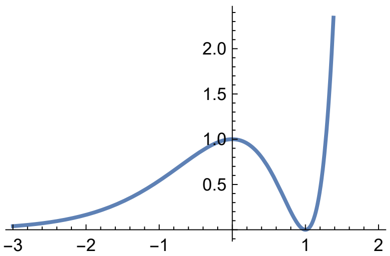
3) $f''(x) = 4e^{2x}\left(x^2 - x\right) + 2e^{2x}(2x-1) = 2e^{2x}\left(2x^2 - 2x + 2x - 1\right) = 2e^{2x}\left(2x^2 - 1\right) = 0$

$\implies x_{1,2} = \pm\frac{1}{\sqrt{2}}$ **(2p)**

x	$x<-\frac{1}{\sqrt{2}}$	$x=-\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}<x<\frac{1}{\sqrt{2}}$	$x=\frac{1}{\sqrt{2}}$	$x>\frac{1}{\sqrt{2}}$
f''	+	0	−	0	+
f	∪	infl.	∩	infl.	∪

(3p)

The graph of f: **(3p)**



$\implies$ the range of f is $[0, \infty)$

5. (10+10 points) Calculate the following integrals:

a) $I_1 = \int_0^{\ln 2} (2x+3)e^{-x}\,dx$ b) $I_2 = \int \frac{1}{x\sqrt{x} + 4\sqrt{x}}\,dx$ (substitution: $t = \sqrt{x}$)

Solution: a) We use the integration by parts method: $\int f' \cdot g = f \cdot g - \int f \cdot g'$

- $f'(x) = e^{-x} \quad \Longrightarrow f(x) = -e^{-x}$
- $g(x) = 2x + 3 \quad \Longrightarrow g'(x) = 2$

$$\Longrightarrow \int (2x+3)\, e^{-x}\, dx = -e^{-x}(2x+3) - \int -e^{-x} \cdot 2\, dx = -e^{-x}(2x+3) - 2e^{-x} + c \quad \textbf{(6p)}$$

$$\Longrightarrow I_1 = \int_0^{\ln 2} (2x+3)\, e^{-x}\, dx = [-e^{-x}(2x+3) - 2e^{-x}]_0^{\ln 2} \quad \textbf{(2p)}$$

$$= \left(\left(-e^{-\ln 2}(2\ln 2 + 3) - 2e^{-\ln 2}\right) - (-3-2)\right) = -\frac{1}{2}(2\ln 2 + 3) - 1 + 5 = \frac{5}{2} - \ln 2 \quad \textbf{(2p)}$$

b) $I_2 = \int \frac{1}{x\sqrt{x} + 4\sqrt{x}}\, dx = ?$ Substitution: $t = \sqrt{x} \Longrightarrow x = x(t) = t^2 \Longrightarrow x'(t) = \frac{dx}{dt} = 2t \Longrightarrow dx = 2t\, dt$

$$\Longrightarrow I_2 = \int \frac{1}{x\sqrt{x} + 4\sqrt{x}}\, dx = \int \frac{1}{t^3 + 4t} \cdot 2t\, dt \ \textbf{(5p)} = \int \frac{2}{t^2+4}\, dt = \frac{2}{4} \int \frac{1}{\left(\frac{t}{2}\right)^2 + 1}\, dt =$$

$$= \frac{1}{2} \cdot \frac{\arctan\left(\frac{t}{2}\right)}{\frac{1}{2}} + c = \arctan\left(\frac{t}{2}\right) + c = \arctan\left(\frac{\sqrt{x}}{2}\right) + c \quad \textbf{(5p)}$$

6. (10+10 points) Calculate the following integrals:

a) $I_3 = \int \frac{x+2}{(x-1)(x^2+2)}\, dx$ b) $I_4 = \int \frac{e^{2x}}{e^{2x} - 5e^x + 6}\, dx$ (substitution: $t = e^x$)

Solution. a) We use partial fraction decomposition:

$\frac{x+2}{(x-1)(x^2+2)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+2}$ **(2p)** Multiplying by $(x-1)(x^2+2)$ we get:

$$x + 2 = A(x^2+2) + (x-1)(Bx+C)$$

$x = 1 \Longrightarrow \quad 3 = 3A + 0 \quad \Longrightarrow A = 1$

$x = 0 \Longrightarrow \quad 2 = 2A - C \quad \Longrightarrow C = 0$ **(3p)**

$x = 2 \Longrightarrow \quad 4 = 6A + 2B + C \quad \Longrightarrow B = 2 - 3A \Longrightarrow B = -1$

$$\Longrightarrow I_3 = \int \frac{x+2}{(x-1)(x^2+2)}\, dx = \int \left(\frac{1}{x-1} - \frac{x}{x^2+2}\right) dx = \int \left(\frac{1}{x-1} - \frac{1}{2}\frac{2x}{x^2+2}\right) dx =$$

$$= \ln|x-1| - \frac{1}{2}\ln(x^2+2) + c \quad \textbf{(5p)}$$

b) $I_4 = \int \frac{e^{2x}}{e^{2x} - 5e^x + 6}\, dx = ?$ (substitution: $t = e^x$)

Substitution: $t = e^x \Longrightarrow x = x(t) = \ln t \Longrightarrow x'(t) = \frac{dx}{dt} = \frac{1}{t} \Longrightarrow dx = \frac{1}{t}\, dt$

$$\Longrightarrow I_4 = \int \frac{e^{2x}}{e^{2x} - 5e^x + 6}\, dx = \int \frac{t^2}{t^2 - 5t + 6} \cdot \frac{1}{t}\, dt = \int \frac{t}{(t-2)(t-3)}\, dt \quad \textbf{(4p)}$$

Partial fraction decomposition: $\frac{t}{(t-2)(t-3)} = \frac{A}{t-2} + \frac{B}{t-3}$

$\Longrightarrow \quad t = A(t-3) + B(t-2)$

$t = 2 \quad \Longrightarrow \quad 2 = -A + 0 \quad \Longrightarrow A = -2$

$t = 3 \quad \Longrightarrow \quad 3 = 0 + B \quad \Longrightarrow B = 3$ **(3p)**

$\Longrightarrow I_4 = \int \left(\frac{-2}{t-2} + \frac{3}{t-3} \right) dt = -2 \ln | t-2 | + 3 \ln | t-3 | + c = -2 \ln | e^x - 2 | + 3 \ln | e^x - 3 | + c$ **(3p)**

7. (10 points) Consider the function $f(x) = \sqrt{\sin^3 x}$ on the interval $x \in [0, \pi]$. Rotate it around the x-axis and find the volume of the arising body.

Solution. The volume is $V = \pi \int_0^\pi f^2(x)\, dx = \pi \int_0^\pi \sin^3 x\, dx$ **(2p)**

$= \pi \int_0^\pi \sin x \cdot \sin^2 x\, dx = \pi \int_0^\pi \sin x (1 - \cos^2 x)\, dx = \pi \int_0^\pi (\sin x + (-\sin x) \cos^2 x)\, dx$ **(2p)**

$= \pi \left[-\cos x + \frac{\cos^3 x}{3} \right]_0^\pi$ **(2p)** $= \pi \left(\left(-\cos \pi + \frac{\cos^3 \pi}{3} \right) - \left(-\cos 0 + \frac{\cos^3 0}{3} \right) \right)$ **(2p)**

$= \pi \left(\left(1 - \frac{1}{3} \right) - \left(-1 + \frac{1}{3} \right) \right) = \frac{4\pi}{3}$ **(2p)**

8.* (10 points - BONUS)

The set of limit points of the set $A \subset \mathbb{R}$ is denoted by A'. Decide whether the following statements are true or false. In each case give a reason for your answer.

a) $A \subset A'$ for all $A \subset \mathbb{R}$.

b) $A' \subset A$ for all $A \subset \mathbb{R}$.

c) There exists $A \subset \mathbb{R}$ such that $A \neq \emptyset$ and $A' = \emptyset$.

d) There exists a bounded set $A \subset \mathbb{R}$ such that A has infinitely many elements and $A' = \emptyset$.

Solution.

a) False. For example, let $A = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$. Then $A' = \{0\}$, so A is not a subset of A'. **(2p)**

b) False. For example, if $A = [0, 1] \cap \mathbb{Q}$ then $A' = [0, 1]$, which is not a subset of A. **(3p)**

c) True. For example, if $A = \{1, 2, 3\}$ then $A' = \emptyset$. **(2p)**

d) False. Let (a_n) be a sequence such that $a_n \in A$ for all n. Since (a_n) is bounded, then by the Bolzano-Weierstrass theorem, (a_n) has a convergent subsequence. The limit of this subsequence is a limit point of the set A, so $A \neq \emptyset$. **(3p)**