Calculus 1, Final exam, Part 2

12th January, 2022

Name: ________________________________ Neptun code: _________

1.	2.	3.	4.	5.	6.	7.	8.	9.	10.	$\sum$

1. (10 points) Find the limit of the following sequence: $a_n = \sqrt[n]{\dfrac{7^n + 5^n}{n^3 + 2}}$

2. (10 points) Decide whether the following series converges or diverges: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n!}\left(\frac{n}{4}\right)^n$

3. (10 points) Find the following limit: $\displaystyle\lim_{x\to 0} \frac{x^3 - \sin(x^2)}{\arctan(x^2)}$

4. (10 points) Calculate the minimum and the maximum of the function $f(x) = x^2 e^{-5x^2 - 8x + 1}$ on the interval $[-2, 0]$.

5. (12 points) Find the Taylor series of the function $f(x) = \sqrt[3]{1 - 2x^3}$ about $x_0 = 0$ and find the interval of convergence.

6. (12 points) Calculate the following integral with the substitution $t = \tan x$: $\displaystyle\int \frac{1}{1 + \tan x}\,dx$

7. (12 points) Calculate the following integral: $\displaystyle\int_0^1 x^3 \ln x \,dx$

8. (12 points) Calculate the following integral: $\displaystyle\int_1^{\infty} \frac{1}{x^2 + 5x + 6}\,dx$

9. (12 points) Consider the function $f(x) = \dfrac{\sqrt{\sin x}}{\cos x}$ on the interval $x \in \left[0, \dfrac{\pi}{4}\right]$. Rotate it around the x-axis and find the volume of the arising body.

10.* (10 points - BONUS) Denote by $\{a\}$ the fractional part of the number $a \in \mathbb{R}$ and let $f(x) = x \cdot \left\{\dfrac{1}{x}\right\}$.

Calculate the following limits:

a) $\displaystyle\lim_{x\to 0} \left\{\frac{1}{x}\right\}$ b) $\displaystyle\lim_{x\to 0} f(x)$ c) $\displaystyle\lim_{x\to +\infty} f(x)$ d) $\displaystyle\lim_{x\to -\infty} f(x)$

Solutions

1. (10 points) Find the limit of the following sequence: $a_n = \sqrt[n]{\frac{7^n + 5^n}{n^3 + 2}}$

Upper estimation:

$$a_n = \sqrt[n]{\frac{7^n + 5^n}{n^3 + 2}} \leq \sqrt[n]{\frac{7^n + 7^n}{0 + 2}} = \sqrt[n]{\frac{2 \cdot 7^n}{2}} = \sqrt[n]{7^n} = 7 \longrightarrow 7$$

Lower estimation:

$$a_n = \sqrt[n]{\frac{7^n + 5^n}{n^3 + 2}} \geq \sqrt[n]{\frac{7^n + 0}{n^3 + n^3}} = \sqrt[n]{\frac{7^n}{2 \cdot n^3}} = \frac{7}{\sqrt[n]{2} \cdot \left(\sqrt[n]{n}\right)^3} \longrightarrow \frac{7}{1 \cdot 1^3} = 7$$

By the sandwich theorem $a_n \longrightarrow 7$.

2. (10 points) Decide whether the following series converges or diverges: $\sum_{n=1}^{\infty} \frac{1}{n!} \left(\frac{n}{4}\right)^n$

By the ratio test: $\frac{|a_{n+1}|}{|a_n|} = \frac{1}{(n+1)!} \frac{(n+1)^{n+1}}{4^{n+1}} \cdot \frac{n! \cdot 4^n}{n^n} = \frac{1}{4} \cdot \frac{1}{n+1} \cdot \frac{(n+1)^{n+1}}{n^n} =$

$$= \frac{1}{4} \cdot \frac{(n+1)^n}{n^n} = \frac{1}{4} \cdot \left(1 + \frac{1}{n}\right)^n \longrightarrow \frac{e}{4} < 1 \Longrightarrow \text{the series converges}$$

3. (10 points) Find the following limit: $\lim_{x \to 0} \frac{x^3 - \sin(x^2)}{\arctan(x^2)}$

The limit has the form $\frac{0}{0}$. By L'Hospital's rule: $\lim_{x \to 0} \frac{3x^2 - 2x\cos(x^2)}{\frac{2x}{1+x^2}} =$

$$= \lim_{x \to 0} \frac{3x - 2\cos(x^2)}{\frac{2}{1+x^2}} = \frac{0 - 2}{2} = -1$$

4. (10 points) Calculate the minimum and the maximum of the function $f(x) = x^2 e^{-5x^2 - 8x + 1}$ on the interval $[-2, 0]$.

$$f'(x) = 2x e^{-5x^2 - 8x + 1} + x^2(-10x - 8) e^{-5x^2 - 8x + 1} = -2x(5x^2 + 4x - 1) e^{-5x^2 - 8x + 1} = 0$$

if $x_1 = 0$ or $x = \frac{-4 \pm \sqrt{16 + 20}}{10} = \frac{-4 \pm 6}{10} \Longrightarrow x_2 = -1$ or $x_3 = \frac{1}{5}$.

Then $x_1, x_2 \in [-2, 0]$ but $x_3 \notin [0, -2]$.

We have to calculate the function values at the critical points x_1, x_2 and at the endpoints of the interval.

$f(x_1) = f(0) = 0$

$f(x_2) = f(-1) = e^4$

$f(-2) = 4e^{-3}$

Since $f(0) < f(-2) < f(-1)$ then the minimum of f is $f(0) = 0$ and the maximum of f is $f(-1) = e^4$ on the interval $[-2, 0]$.

5. (12 points) Find the Taylor series of the function $f(x) = \sqrt[3]{1-2x^3}$ about $x_0 = 0$ and find the interval of convergence.

Using the formula for the binomial series: $(1+u)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} u^k$ where $|u| < 1 = R$,

we get that

$$f(x) = \sqrt[3]{1-2x^3} = \left(1 + \left(-2x^3\right)\right)^{\frac{1}{3}} = \sum_{k=0}^{\infty} \binom{\frac{1}{3}}{k} \left(-2x^3\right)^k = \sum_{k=0}^{\infty} \binom{\frac{1}{3}}{k} (-2)^k x^{3k}$$

where $|u| = |-2x^3| < 1 \Longrightarrow |x| < \frac{1}{\sqrt[3]{2}}$.

6. (12 points) Calculate the following integral with the substitution $t = \tan x$: $\int \frac{1}{1+\tan x} \mathrm{dx}$

$$t = \tan x \Longrightarrow x = x(t) = \arctan t \Longrightarrow x'(t) = \frac{\mathrm{dx}}{\mathrm{dt}} = \frac{1}{1+t^2} \Longrightarrow \mathrm{dx} = \frac{1}{1+t^2} \mathrm{dt}$$

$$I = \int \frac{1}{1+\tan x} \mathrm{dx} = \int \frac{1}{1+t} \cdot \frac{1}{1+t^2} \mathrm{dt}$$

Partial fraction decomposition: $\frac{1}{(1+t)\cdot\left(1+t^2\right)} = \frac{A}{1+t} + \frac{Bt+C}{1+t^2} = \frac{A\left(1+t^2\right) + (Bt+C)(1+t)}{(1+t)\cdot\left(1+t^2\right)}$

$$\Longrightarrow \quad 1 = A\left(1+t^2\right) + (Bt+C)(1+t)$$

$$t = -1 \Longrightarrow \quad 1 = 2A + 0 \Longrightarrow A = \frac{1}{2}$$

$$t = 0 \Longrightarrow \quad 1 = A + C \Longrightarrow C = 1 - A = \frac{1}{2}$$

$$t = 1 \Longrightarrow \quad 1 = 2A + 2B + 2C \Longrightarrow 2B = 1 - 2A - 2C = 1 - 1 - 1 = -1 \Longrightarrow B = -\frac{1}{2}$$

$$\frac{1}{(1+t)\cdot\left(1+t^2\right)} = \frac{1}{2}\left(\frac{1}{t+1} + \frac{-t+1}{t^2+1}\right)$$

The integral is:

$$I = \int \frac{1}{2}\left(\frac{1}{t+1} + \frac{-t+1}{t^2+1}\right)\mathrm{dt} = \int \frac{1}{2}\left(\frac{1}{t+1} - \frac{1}{2}\frac{2t}{t^2+1} + \frac{+1}{t^2+1}\right)\mathrm{dt} =$$

$$= \frac{1}{2}\left(\ln|t+1| - \frac{1}{2}\ln\left(t^2+1\right) + \arctan t\right) + c = \frac{1}{2}\ln(\tan x + 1) - \frac{1}{4}\ln\left(\tan^2 x + 1\right) + \frac{1}{2}x + c$$

7. (12 points) Calculate the following integral: $\int_0^1 x^3 \ln x \, \mathrm{dx}$

With integration by parts: $\int x^3 \ln x \, \mathrm{dx} = -\frac{x^4}{16} + \frac{1}{4} x^4 \ln x + c$

The definite integral is an improper integral:

$$\int_0^1 x^3 \ln x \, \mathrm{dx} = \lim_{\varepsilon \to 0+} \int_\varepsilon^1 x^3 \ln x \, \mathrm{dx} = \lim_{\varepsilon \to 0+} \left[-\frac{x^4}{16} + \frac{1}{4} x^4 \ln x\right]_\varepsilon^1 = \lim_{\varepsilon \to 0+}\left(\left(-\frac{1}{16} + 0\right) - \left(-\frac{\varepsilon^4}{16} + \frac{1}{4}\varepsilon^4 \ln \varepsilon\right)\right) =$$

$$= \left(-\frac{1}{16} + 0\right) - (-0 + 0) = -\frac{1}{16}$$

By L'Hospital's rule: $\lim\limits_{x\to 0+} x^4 \ln x = \lim\limits_{x\to 0+} \dfrac{\ln x}{\frac{1}{x^4}} = \lim\limits_{x\to 0+} \dfrac{\frac{1}{x}}{-4\cdot\frac{1}{x^5}} = \lim\limits_{x\to 0+} \dfrac{x^4}{-4} = 0.$

8. (12 points) Calculate the following integral: $\displaystyle\int_1^\infty \frac{1}{x^2+5x+6}\,dx$

Partial fraction decomposition: $\dfrac{1}{x^2+5x+6} = \dfrac{1}{(x+2)(x+3)} = \dfrac{A}{x+2} + \dfrac{B}{x+3} =\ldots= \dfrac{1}{x+2} - \dfrac{1}{x+3}$

$$\int_1^\infty \frac{1}{x^2+5x+6}\,dx = \lim_{A\to\infty}\int_1^A \frac{1}{x^2+5x+6}\,dx = \lim_{A\to\infty}[\ln|x+2| - \ln|x+3|]_1^A =$$

$$= \lim_{A\to\infty}(\ln(A+2) - \ln(A+3) - (\ln 3 - \ln 4)) = \lim_{A\to\infty}\left(\ln\frac{A+2}{A+3} - \ln\frac{3}{4}\right) = \ln 1 - \ln\frac{3}{4} = \ln\frac{4}{3}$$

9. (12 points) Consider the function $f(x) = \dfrac{\sqrt{\sin x}}{\cos x}$ on the interval $x \in \left[0, \dfrac{\pi}{4}\right]$. Rotate it around the x-axis and find the volume of the arising body.

The volume is $V = \pi\displaystyle\int_0^{\pi/4} f^2(x)\,dx = \pi\int_0^{\pi/4}\frac{\sin x}{\cos^2 x}\,dx = \pi\int_0^{\pi/4} -(-\sin x)(\cos x)^{-2}\,dx =$

$$= \pi\left[-\frac{(\cos x)^{-1}}{-1}\right]_0^{\pi/4} = \pi\left[\frac{1}{\cos x}\right]_0^{\pi/4} = \pi\left(\frac{1}{\frac{\sqrt{2}}{2}} - 1\right) = \pi\left(\sqrt{2} - 1\right)$$

10.* (10 points - BONUS) Denote by $\{a\}$ the fractional part of the number $a \in \mathbb{R}$ and let $f(x) = x\cdot\left\{\dfrac{1}{x}\right\}$. Calculate the following limits:

a) $\lim\limits_{x\to 0}\left\{\dfrac{1}{x}\right\}$ b) $\lim\limits_{x\to 0} f(x)$ c) $\lim\limits_{x\to +\infty} f(x)$ d) $\lim\limits_{x\to -\infty} f(x)$

a) Using the sequential criterion for the limit, it can be shown that the limit doesn't exist.
For example, let $x_n = \dfrac{1}{n}$ and $y_n = \dfrac{1}{n+\frac{1}{2}}$. Then $x_n \longrightarrow 0$ and $y_n \longrightarrow 0$ but

$$\left\{\frac{1}{x_n}\right\} = \{n\} = 0 \neq \left\{\frac{1}{y_n}\right\} = \left\{n + \frac{1}{2}\right\} = \frac{1}{2}$$

b) Since the range of the function $g(x) = \{x\}$ is $[0, 1)$ and $\lim\limits_{x\to 0} x = 0$ then $\lim\limits_{x\to 0} f(x) = 0$.

c) If $x > 1$ then $\dfrac{1}{x} \in (0, 1)$, so $\left\{\dfrac{1}{x}\right\} = \dfrac{1}{x} \implies f(x) = x\cdot\dfrac{1}{x} = 1 \implies \lim\limits_{x\to +\infty} f(x) = 1$.

d) If $x < -1$ then $\dfrac{1}{x} \in (-1, 0)$, so $\left\{\dfrac{1}{x}\right\} = 1 + \dfrac{1}{x} \implies f(x) = x\cdot\left(1 + \dfrac{1}{x}\right) = x + 1 \longrightarrow -\infty$ if $x \to -\infty$.