Calculus 1, Final exam, Part 2

10th January, 2023

Name: ____________________________________ Neptun code: _______________

1.: _____ 2.: _____ 3.: _____ 4.: _____ 5.: _____ 6.: _____ 7.: _____ 8.: _____ Sum: _____

1. (10 points) Calculate the following limit: $\lim\limits_{x \to 0} \dfrac{x^3 - \arctan(x^2)}{\sin(x^2)}$

2. (5+5+5 points) Calculate the derivatives of the following functions:

a) $f : (0, \infty) \longrightarrow \mathbb{R},\ \ f(x) = (\sin 2x)^{\frac{1}{x}}$

b) $f(x) = \sqrt{\dfrac{e^{2x} \cdot \cos x}{\ln(x^2 + 1)}}$

c) $f(x) = \displaystyle\int_0^{x^3} \sqrt{e^t + t^2}\ \mathrm{dt}$

3. (10 points) What is the largest possible volume of a right circular cylinder without top that can be made from 48 square decimeters of metal?

4. (15 points) Analyze the following function and sketch its graph: $f(x) = x + 2 + \dfrac{2}{x - 1}$.

5. (10+10 points) Calculate the following integrals:

a) $I_1 = \displaystyle\int x \arcsin(x^2)\, \mathrm{dx}$

b) $I_2 = \displaystyle\int_0^3 \frac{\sqrt{x}}{x + 1}\, \mathrm{dx}$ (substitution: $t = \sqrt{x}$)

6. (10+10 points) Calculate the following integrals:

a) $I_3 = \displaystyle\int \frac{x + 1}{4x^3 + x}\, \mathrm{dx}$

b) $I_4 = \displaystyle\int \frac{e^x}{e^{2x} + 4e^x + 3}\, \mathrm{dx}$ (substitution: $t = e^x$)

7. (10 points) Calculate the area of the region enclosed by the curves $f(x) = x^2 - 3x + 2$ and $g(x) = 5 - x$.

8.* (10 points - BONUS)

How many real roots does the function $F(x) = \displaystyle\int_{-1}^{x} \sinh(t^3)\, \mathrm{dt}$ have?

Solutions

1. (10 points) Calculate the following limit: $\lim_{x\to 0}\frac{x^3-\arctan(x^2)}{\sin(x^2)}$

Solution. The limit has the form $\frac{0}{0}$, so the L'Hospital's rule can be applied:

$$\lim_{x\to 0}\frac{x^3-\arctan(x^2)}{\sin(x^2)} \overset{\frac{0}{0},L'H}{=} \lim_{x\to 0}\frac{3x^2-\frac{2x}{1+x^4}}{\cos(x^2)\cdot 2x}\ \textbf{(4p)}\ \overset{\frac{0}{0},L'H}{=} \lim_{x\to 0}\frac{6x-\frac{2(1+x^4)-2x\cdot 4x^3}{(1+x^4)^2}}{-\sin(x^2)\cdot 2x\cdot 2x+\cos(x^2)\cdot 2}\ \textbf{(4p)}\ = \frac{0-2}{0+2}=-1\ \textbf{(2p)}$$

Or: $\lim_{x\to 0}\frac{3x^2-\frac{2x}{1+x^4}}{\cos(x^2)\cdot 2x}=\lim_{x\to 0}\frac{3x-\frac{2}{1+x^4}}{\cos(x^2)\cdot 2}=\frac{0-2}{0}=-2$

2. (5+5+5 points) Calculate the derivatives of the following functions:

a) $f:(0,\infty)\longrightarrow\mathbb{R},\ \ f(x)=(\sin 2x)^{\frac{1}{x}}$

b) $f(x)=\sqrt{\frac{e^{2x}\cdot\cos x}{\ln(x^2+1)}}$

c) $f(x)=\int_0^{x^3}\sqrt{e^t+t^2}\ dt$

Solution.

a) $f(x)=(\sin 2x)^{\frac{1}{x}}=e^{\ln\left((\sin 2x)^{\frac{1}{x}}\right)}=e^{\frac{1}{x}\ln(\sin 2x)}$

$\Longrightarrow f'(x)=e^{\frac{1}{x}\ln(\sin 2x)}\cdot\left(\frac{1}{x}\ln(\sin 2x)\right)'=(\sin 2x)^{\frac{1}{x}}\cdot\left(-\frac{1}{x^2}\ln(\sin 2x)+\frac{1}{x}\cdot\frac{1}{\sin 2x}\cdot\cos 2x\cdot 2\right)$

b) $f(x)=\sqrt{\frac{e^{2x}\cdot\cos x}{\ln(x^2+1)}}\ \Longrightarrow\ f'(x)=\frac{1}{2}\left(\frac{e^{2x}\cdot\cos x}{\ln(x^2+1)}\right)^{-\frac{1}{2}}\cdot\frac{(e^{2x}\cdot 2\cdot\cos x+e^{2x}\cdot(-\sin x))\cdot\ln(x^2+1)-e^{2x}\cdot\cos x\cdot\frac{2x}{x^2+1}}{(\ln(x^2+1))^2}$

c) $f(x)=\int_0^{x^3}\sqrt{e^t+t^2}\ dt=g(x^3)$ where $g(x)=\int_0^x\sqrt{e^t+t^2}\ dt$

Since the integrand $\sqrt{e^t+t^2}$ is a continuous function of t then g is differentiable and $g'(x)=\sqrt{e^x+x^2}$

$\Longrightarrow f'(x)=g'(x^3)\cdot 3x^2=\sqrt{e^{x^3}+(x^3)^2}\cdot 3x^2=\sqrt{e^{x^3}+x^6}\cdot 3x^2$

3. (10 points) What is the largest possible volume of a right circular cylinder without top that can be made from 48 square decimeters of metal?

Solution. Let the radius of the base circle be r and the height of the cylinder by h.
We have to minimize the volume $V=r^2\pi h$. The surface area is $A=r^2\pi+2r\pi h=48$ from where $h=\frac{48-r^2\pi}{2\pi r}$, so the volume as a function of r is $V(r)=r^2\pi\frac{48-r^2\pi}{2\pi r}=\frac{1}{2}(48r-r^3\pi)$. **(4p)** Then $V'(r)=\frac{\pi}{2}(48-3\pi r^2)=0$ from where $r=\frac{4}{\sqrt{\pi}}$ (since $r>0$). **(3p)**

$V''(r) = \frac{\pi}{2} \cdot (-6\pi r)$ and thus $V''\left(\frac{4}{\sqrt{\pi}}\right) < 0$, so V has a local maximum for $r = \frac{4}{\sqrt{\pi}}$. **(2p)**

The maximum of the volume is $V = \frac{64}{\sqrt{\pi}}$. **(1p)**

4. (15 points) Analyze the following function and sketch its graph: $f(x) = x + 2 + \frac{2}{x-1}$.

Solution.

1) The domain of f is $D_f = \mathbb{R} \setminus \{1\}$.

The zeros of f are: $x + 2 + \frac{2}{x-1} = 0 \implies x_1 = 0,\ x_2 = -1$

The limits of f at $\pm\infty$ and at 1 ± 0 are:

$\lim_{x\to\infty} f(x) = \infty,\ \lim_{x\to-\infty} f(x) = -\infty,\ \lim_{x\to 1+0} f(x) = \infty,\ \lim_{x\to 1-0} f(x) = -\infty$ **(2p)**

2) $f'(x) = 1 - \frac{2}{(x-1)^2} = 0 \implies (x-1)^2 = 2 \implies x = 1 \pm \sqrt{2}$ **(5p)**

x	$x<1-\sqrt{2}$	$x=1-\sqrt{2}$	$1-\sqrt{2}<x<1$	x=1	$1<x<1+\sqrt{2}$	$x=1+\sqrt{2}$	$1+\sqrt{2}<x$
f'	+	0	−	not def.	−	0	+
f	↗	loc. max.	↘	not def.	↘	loc. min.	↗

$\left(f\left(1-\sqrt{2}\right) = 3 - 2\sqrt{2},\ \ f\left(1+\sqrt{2}\right) = 3 + 2\sqrt{2}\right)$

3) $f''(x) = \frac{4}{(x-1)^3} \neq 0$ **(5p)**

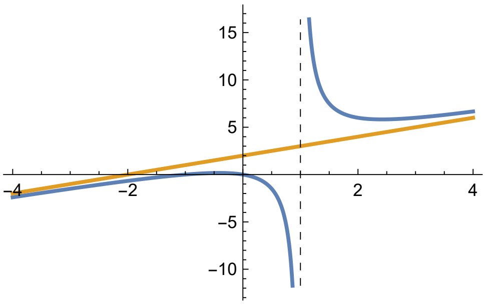
x	x<1	x=1	x>1
f''	−	not def.	+
f	∩	not def.	∪

Linear asymptote of f: $y = Ax + B$, where $A = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{x + 2 + \frac{2}{x-1}}{x} = 1$ and

$B = \lim_{x\to\infty} (f(x) - Ax) = \lim_{x\to\infty} \left(2 + \frac{2}{x-1}\right) = 2.$

$\implies y = x + 2$

The graph of f: **(3p)**



$\implies$ the range of f is $\left(-\infty, 3 - 2\sqrt{2}\right] \cup \left[3 + 2\sqrt{2}, \infty\right)$

5. (10+10 points) Calculate the following integrals:

a) $I_1 = \int x \arcsin(x^2)\, dx$ b) $I_2 = \int_0^3 \frac{\sqrt{x}}{x+1}\, dx$ (substitution: $t = \sqrt{x}$)

Solution: a) We use the integration by parts method: $\int f' \cdot g = f \cdot g - \int f \cdot g'$

- $f'(x) = x \implies f(x) = \frac{x^2}{2}$
- $g(x) = \arcsin(x^2) \implies g'(x) = \frac{1}{\sqrt{1-x^4}} \cdot 2x$

$$\implies I_1 = \int x \arcsin(x^2)\, dx = \frac{x^2}{2} \cdot \arcsin(x^2) - \int \frac{x^2}{2} \cdot \frac{1}{\sqrt{1-x^4}} \cdot 2x\, dx =$$

$$= \frac{x^2}{2} \cdot \arcsin(x^2) - \int \frac{x^3}{\sqrt{1-x^4}}\, dx =$$

$$= \frac{x^2}{2} \cdot \arcsin(x^2) + \int \frac{1}{4} \cdot (-4x^3)(1-x^4)^{-\frac{1}{2}}\, dx = \frac{x^2}{2} \cdot \arcsin(x^2) + \frac{1}{4} \cdot \frac{(1-x^4)^{\frac{1}{2}}}{\frac{1}{2}} + c =$$

$$= \frac{x^2}{2} \cdot \arcsin(x^2) + \frac{1}{2} \cdot \sqrt{1-x^4} + c$$

b) $I_2 = \int_0^3 \frac{\sqrt{x}}{x+1}\, dx = ?$ Substitution: $t = \sqrt{x} \implies x = x(t) = t^2 \implies x'(t) = \frac{dx}{dt} = 2t \implies dx = 2t\, dt$

The bounds will change: $x_1 = 0 \implies t_1 = \sqrt{0} = 0$

$x_2 = 3 \implies t_2 = \sqrt{3}$

$$\implies I_2 = \int_0^3 \frac{\sqrt{x}}{x+1}\, dx = \int_0^{\sqrt{3}} \frac{t}{t^2+1} \cdot 2t\, dt \;\textbf{(5p)} = \int_0^{\sqrt{3}} \frac{(2t^2+2)-2}{t^2+1}\, dt = \int_0^{\sqrt{3}} \left(2 - \frac{2}{t^2+1}\right) dt =$$

$$= [2t - 2\arctan(t)]_0^{\sqrt{3}} = \left(2\sqrt{3} - 2\arctan\left(\sqrt{3}\right)\right) - (0-0) \;\textbf{(5p)}\; = 2\left(\sqrt{3} - \frac{\pi}{3}\right)$$

6. (10+10 points) Calculate the following integrals:

a) $I_3 = \int \frac{x+1}{4x^3+x}\, dx$ b) $I_4 = \int \frac{e^x}{e^{2x}+4e^x+3}\, dx$ (substitution: $t = e^x$)

Solution. a) We use partial fraction decomposition:

$\frac{x+1}{4x^3+x} = \frac{x+1}{x(4x^2+1)} = \frac{A}{x} + \frac{Bx+C}{4x^2+1}$ **(2p)** Multiplying by $x(4x^2+1)$ we get:

$$x + 1 = A(4x^2+1) + x(Bx+C)$$

$x = 0 \implies 1 = A + 0$

$x = 1 \implies 2 = 5A + B + C \implies 2C = 2 \implies C = 1,\; B = -4$ **(3p)**

$x = -1 \implies 0 = 5A + B - C$

$$\Longrightarrow I_3 = \int \frac{x+1}{4x^3+x}\,dx = \int\left(\frac{A}{x} + \frac{Bx+C}{4x^2+1}\right)dx = \int\left(\frac{1}{x} + \frac{-4x+1}{4x^2+1}\right)dx =$$

$$= \int\left(\frac{1}{x} + \left(-\frac{1}{2}\right)\frac{8x}{4x^2+1} + \frac{1}{(2x)^2+1}\right)dx = \ln|x| - \frac{1}{2}\ln(4x^2+1) + \frac{1}{2}\arctan(2x) + c \ \textbf{(5p)}$$

b) $I_4 = \int \frac{e^x}{e^{2x}+4e^x+3}\,dx = ?$ (substitution: $t = e^x$)

Substitution: $t = e^x \Longrightarrow x = x(t) = \ln t \Longrightarrow x'(t) = \frac{dx}{dt} = \frac{1}{t} \Longrightarrow dx = \frac{1}{t}\,dt$

$$\Longrightarrow I_4 = \int \frac{e^x}{e^{2x}+4e^x+3}\,dx = \int \frac{t}{t^2+4t+3}\cdot\frac{1}{t}\,dt = \int \frac{1}{(t+1)(t+3)}\,dt \ \textbf{(4p)}$$

Partial fraction decomposition: $\frac{1}{(t+1)(t+3)} = \frac{A}{t+1} + \frac{B}{t+3}$

$\Longrightarrow \quad 1 = A(t+3) + B(t+1)$

$t = -1 \Longrightarrow \quad 1 = 2A + 0 \Longrightarrow A = \frac{1}{2}$

$t = -3 \Longrightarrow \quad 1 = 0 - 2B \Longrightarrow B = -\frac{1}{2}$ **(3p)**

$$\Longrightarrow I_4 = \int\left(\frac{1}{2}\frac{1}{t+1} - \frac{1}{2}\frac{1}{t+3}\right)dt = \frac{1}{2}\ln|t+1| - \frac{1}{2}\ln|t+3| + c = \frac{1}{2}\ln(e^x+1) - \frac{1}{2}\ln(e^x+3) + c \ \textbf{(3p)}$$

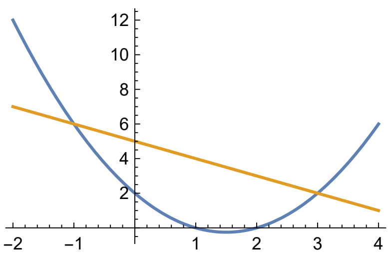
$$= \frac{1}{2}\ln\frac{e^x+1}{e^x+3} + c$$

7. (10 points) Calculate the area of the region enclosed by the curves $f(x) = x^2 - 3x + 2$ and $g(x) = 5 - x$.

Results:

$x^2 - 3x + 2 = 5 - x \Longrightarrow$ the x coordinate of the intersections points of the curves are $x_1 = -1$ and $x_2 = 3$.

The area is $\int_{-1}^{3}((5-x) - (x^2 - 3x + 2))\,dx$ **(5p)** $= \left[3x + x^2 - \frac{x^3}{3}\right]_{-1}^{3} = \frac{32}{3}$ **(5p)**



8.* (10 points - BONUS)

How many real roots does the function $F(x) = \int_{-1}^{x} \sinh(t^3)\,dt$ have?

Solution. $F(-1) = 0$. **(1p)** Since $F'(x) = \sinh(x^3)$ **(2p)** and $F'(x) = 0 \Longleftrightarrow x = 0$ **(1p)** then F may have at most one negative and one positive real root. **(2p)** Since $F(0) < 0$ and $\lim_{x\to\infty} F(x) = \infty$ then F has a positive real root. **(4p)**