Calculus 1, Final exam, Part 2

5th January, 2022

Name: ___________________________________ Neptun code: __________

1.	2.	3.	4.	5.	6.	7.	8.	9.	10.	$\sum$

1. (10 points) Find the limit of the following sequence: $a_n = \sqrt[n]{\left(\frac{2n}{3n+1}\right)^{n+1}}$

2. (10 points) Decide whether the following series converges or diverges: $\sum_{n=1}^{\infty} \frac{(n+3)!}{\sqrt{(2n)!}}$

3. (10 points) Find the following limit: $\lim_{x \to \frac{1}{3}} \left(\frac{1}{\ln 3x} - \frac{1}{3x-1} \right)$

4. (10 points) A rectangle R is inscribed in a semi-circle S of radius 1. What is the minimal possible perimeter of R?

5. (12 points) Give the Taylor polynomial of order 3 of the function $f(x) = x^2 + 5x - 3 + \sin 2x$ around the point $x_0 = 0$. Estimate the error if the value of $f(0.1)$ is approximated by $T_3(0.1)$.

6. (12 points) Calculate the following integral with the substitution $t = e^x$: $\int \frac{e^x + 2}{e^{2x} + 1}\, dx$

7. (12 points) Calculate the following integral: $\int_0^{\infty} e^{-x}(3x+1)\, dx$

8. (12 points) Calculate the following integral: $\int_{-\infty}^{\infty} \frac{1}{(1+x^2)(\arctan x + \pi)}\, dx$

9. (12 points) Find the surface area of the following body of rotation (we rotate the given curve around the x-axis over the given interval):

$y = \sqrt{x+1}, \quad x \in [1, 5]$

10.* (10 points - BONUS) Assume that the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is differentiable everywhere. Prove that if $\lim_{x \to \infty} f'(x) = \infty$ then f is not uniformly continuous on $[0, \infty)$.

Results

1. The limit is $\frac{2}{3}$.

2. $\frac{a_{n+1}}{a_n} \longrightarrow \frac{1}{2} < 1 \implies$ the series is convergent.

3. The limit is $\frac{1}{2}$.

4. The minimal possible perimeter is $2\sqrt{5}$.

5. The Taylor polynomial is $T_3(x) = -3 + 7x + \frac{2}{2!}x^2 - \frac{8}{3!}x^3$

The remainder term is $R_3(x, \xi) = \frac{16\sin 2\xi}{4!}x^4$ where $x_0 = 0 < \xi < x = 0.1$

The error is $|E| = |R_3(0.1, \xi)| = \left|\frac{16\sin 2\xi}{4!}0.1^4\right| \leq \frac{16\cdot 1}{4!}0.1^4$

6. $\int \frac{e^x + 2}{e^{2x} + 1}\,dx = 2x + \arctan(e^x) - \ln(1 + e^{2x}) + c$

7. $\int_0^\infty e^{-x}(3x + 1)\,dx = 4$

8. $\int_{-\infty}^\infty \frac{1}{(1 + x^2)(\arctan x + \pi)}\,dx = \ln 3$

9. $A = 2\pi \int_1^5 \sqrt{x + 1}\sqrt{1 + \frac{1}{4x + 4}}\,dx = \frac{49\pi}{3}$

10. Since $f' \longrightarrow \infty$ then for all $K > 0$ there exists $P(K) > 0$ such that $f'(x) > K$ if $x > P(K)$.
$\implies$ if $x > y > P(K)$ then by Lagrange's mean value theorem there exists $c \in (y, x)$ such that
$\frac{f(x) - f(y)}{x - y} = f'(c) > K \implies f(x) - f(y) > K(x - y)$ and $K(x - y) > 1$ if $K > \frac{1}{x - y}$.