Practice exercises 6.

1. Decide whether the following series are convergent or divergent.

a) $\sum_{n=1}^{\infty} \frac{10^n}{n!}$ b) $\sum_{n=1}^{\infty} \frac{n^4}{2^n}$ c) $\sum_{n=1}^{\infty} \frac{3^n}{n^3\, 2^n}$ d) $\sum_{n=1}^{\infty} \frac{2^n\, n^2}{n!}$

e) $\sum_{n=1}^{\infty} \frac{n}{(\ln n)^n}$ f) $\sum_{n=1}^{\infty} \frac{(n!)^n}{(n^n)^2}$ g) $\sum_{n=1}^{\infty} \frac{(n!)^n}{n^{(n^2)}}$ h) $\sum_{n=1}^{\infty} \frac{n \log n}{2^n}$

i) $\sum_{n=1}^{\infty} \frac{n^n}{(n!)^2}$ j) $\sum_{n=1}^{\infty} \frac{2^n \cdot n!}{n^n}$ k) $\sum_{n=1}^{\infty} \frac{1}{n!} \left(\frac{n}{e-1}\right)^n$ l) $\sum_{n=1}^{\infty} \left(\frac{n}{n^2+1}\right)^{n^2}$

m) $\sum_{n=1}^{\infty} \left(\frac{n+1}{n+2}\right)^{2n^2+1}$ n) $\sum_{n=1}^{\infty} \left(\frac{4n}{4n+1}\right)^{3n^2}$ o) $\sum_{n=1}^{\infty} \frac{(n+2)^n}{(n+1)!}$ p) $\sum_{n=1}^{\infty} \frac{(3n)!}{(n!)^3}$

q) $\sum_{n=1}^{\infty} \frac{4^n\, (n!)^2}{(2n)!}$ r) $\sum_{n=1}^{\infty} \frac{(n!)^2 - 2^n}{(2n)!}$ s) $\sum_{n=1}^{\infty} \frac{n^4 + n \log n - 2^n}{n! + n^{10} + 3^n}$ t) $\sum_{n=1}^{\infty} \frac{(n!)^3}{(3n)!} \left(1 + \frac{1}{n}\right)^{n^2+1}$

2*. Convergent or divergent?

$$\frac{1000}{1} + \frac{1000 \cdot 1001}{1 \cdot 3} + \frac{1000 \cdot 1001 \cdot 1002}{1 \cdot 3 \cdot 5} + \ldots$$

3. Show that the following series are convergent. Estimate the error if the sum of the series is approximated by the 100th partial sum.

a) $\sum_{n=1}^{\infty} (-1)^n \frac{n+1}{n^2+2}$ b) $\sum_{n=1}^{\infty} (-1)^n \frac{5^n}{2^n + 10^n}$

4. Are the following series convergent or divergent?

a) $\sum_{n=1}^{\infty} (-1)^n \frac{\sin\left(\frac{n\pi}{2}\right)}{n}$ b) $\sum_{n=1}^{\infty} (-1)^n \frac{2n-1}{3n}$ c) $\sum_{n=1}^{\infty} \left(\frac{4-n}{4+n}\right)^n$

d) $\sum_{n=2}^{\infty} (-1)^{n+1} \frac{2n}{n^2-1}$ e) $\sum_{n=2}^{\infty} (-1)^n \frac{1}{n - \sqrt{n}}$ f) $\sum_{n=1}^{\infty} (-1)^n \left(\sqrt{n+1} - \sqrt{n}\right)$

g) $\sum_{n=1}^{\infty} (-1)^n \log\left(1 + \frac{1}{n}\right)$ h) $\sum_{n=1}^{\infty} (-1)^n \left(\sqrt[n]{n} - 1\right)$

5. Are the following series absolutely or conditionally convergent?

a) $\sum_{n=1}^{\infty} \frac{(-3)^n}{4^n + n}$ b) $\sum_{n=1}^{\infty} \frac{(-3)^n}{2 + n\, 3^n}$ c) $\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n \sqrt{n}}$

d) $\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{\log n}$ e) $\sum_{n=1}^{\infty} \frac{(-1)^n}{2n^2 - 3n + 8}$

6. Using the product theorem for series evaluate the following sums in closed form:

a) $\sum_{n=1}^{\infty} n\, x^n$ b) $\sum_{n=1}^{\infty} n^2\, x^n$ c) $\sum_{n=1}^{\infty} (n+1)^3\, x^n$

7. Let $f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n!}$. Using the product theorem for series prove that $f(x+y) = f(x) \cdot f(y)$.

8. For what values of x do the following power series converge? In cases a), b), c) evaluate the sum.

a) $\sum_{n=1}^{\infty} x^n$ b) $\sum_{n=1}^{\infty} (x-2)^n$ c) $\sum_{n=1}^{\infty} 3^{n+1}\, x^n$ d) $\sum_{n=1}^{\infty} n\, x^n$

e) $\sum_{n=1}^{\infty} n^n\, x^n$ f) $\sum_{n=1}^{\infty} \frac{1}{n^2}\, x^n$ g) $\sum_{n=1}^{\infty} \frac{x^n}{n!}$ h) $\sum_{n=1}^{\infty} \frac{2^n}{n!}\, x^n$

i) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n\sqrt{n}}\,(x+3)^n$ j) $\sum_{n=1}^{\infty} \cos\left(\frac{n\pi}{4}\right) \frac{n+1}{n^2+1}\, x^n$ k) $\sum_{n=1}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$

l) $\sum_{n=1}^{\infty} \frac{n}{(n+1)\, 3^n}\, x^{2n}$ m) $\sum_{n=1}^{\infty} \frac{n+1}{9^n}\,(x-2)^{2n}$ n)* $\sum_{n=1}^{\infty} \frac{n!}{n^n}\, x^n$