Calculus 1, Midterm Test 2

2nd December, 2021

Name: ___________________________________ Neptun code: __________

1.	2.	3.	4.	5.	6.	7.	8.	9.	$\sum$

1. (10 points) Let $H = \left\{(-1)^n \cdot \dfrac{n-1}{n} : n \in \mathbb{N}\right\} \cup ([3,4] \cap \mathbb{Q}) \cup (5, 8]$.

Find the set of interior points, boundary points, limit points and isolated points of H.

2. (9 points) Determine the points of discontinuity of the function $f(x) = x \arctan \dfrac{1}{x(x+2)}$.

What type of discontinuities are these?

3. (9 points) Find the equation of the tangent line to the function $f(x) = \dfrac{e^{x^2} \ln(2x+5)}{\cos x}$ at $x_0 = 0$.

4. (9+9+9 points) Calculate the following limits:

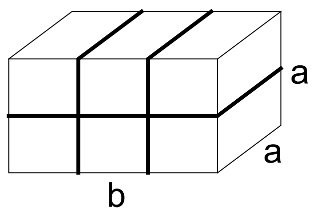
a) $\displaystyle\lim_{x\to 0} \frac{\sqrt{1+8x} - e^{4x}}{x \sin 2x}$ b) $\displaystyle\lim_{x\to 0} (\cos x)^{\frac{1}{\sin^2 x}}$ c) $\displaystyle\lim_{x\to \infty} \frac{\sinh(2x+3)}{\cosh(2x-5)}$

5. (9 points) Find the values of the parameters such that the following function

be differentiable on $\mathbb{R}$: $f(x) = \begin{cases} \dfrac{a}{x^2+1} & \text{if } x \geq 1 \\ b x^4 + 1 & \text{if } x < 1 \end{cases}$

6. (18 points) Analyze the following function and plot its graph: $f(x) = \dfrac{x^2 + x + 1}{x^2 + 1}$.

7. (9 points) The volume of a rectangular box with a square base is $V = 20\ \text{dm}^3$. A rope is tied around the box as shown in the figure. Find the dimensions of the box if the length of the rope is minimal.



8. (9 points) Estimate the value of $\sqrt[3]{1.2}$ by the Taylor polynomial of order 2 of $f(x) = \sqrt[3]{1+x}$ at center 0. Give an upper bound for the error of the approximation.

9.* (10 points - BONUS) Without the help of a calculator, by investigating the function $f(x) = x - e\ln(x)$, show that $e^3 > 3^e$.

Solutions

1. (10 points) Let $H = \left\{(-1)^n \cdot \frac{n-1}{n} : n \in \mathbb{N}\right\} \cup ([3, 4] \cap \mathbb{Q}) \cup (5, 8]$.

Find the set of interior points, boundary points, limit points and isolated points of H.

Solution:

Set of interior points: $\operatorname{int} H = (5, 8)$ **(2p)**

Set of boundary points: $\partial H = \left\{(-1)^n \cdot \frac{n-1}{n} : n \in \mathbb{N}\right\} \cup \{-1, 1\} \cup [3, 4] \cup \{5, 8\}$ **(3p)**

Set of limit points: $H' = \{-1, 1\} \cup [3, 4] \cup [5, 8]$ **(3p)**

Set of isolated points: $\left\{(-1)^n \cdot \frac{n-1}{n} : n \in \mathbb{N}\right\}$ **(2p)**

2. (9 points) Determine the points of discontinuity of the function $f(x) = x \arctan \frac{1}{x(x+2)}$.

What type of discontinuities are these?

Solution. The arctan function is continuous and the composition and ratio of continuous functions is continuous if the denominator is not 0, so the points of discontinuities are $x_1 = 0$ and $x_2 = -2$ **(1p)**.

If $x_1 = 0$: $\lim\limits_{x \to 0} f(x) = 0$ **(1p)**, since the arctan function is bounded **(1p)**

$\Longrightarrow f$ has a removable discontinuity at $x_1 = 0$. **(1p)**

If $x_2 = -2$: $\lim\limits_{x \to -2+} f(x) = \lim\limits_{x \to -2+} x \arctan \frac{1}{x(x+2)} = \lim\limits_{y \to -\infty} (-2) \arctan y = -2 \cdot \left(-\frac{\pi}{2}\right) = \pi$

$\lim\limits_{x \to -2-} f(x) = \lim\limits_{x \to -2-} x \arctan \frac{1}{x(x+2)} = \lim\limits_{y \to \infty} (-2) \arctan y = -2 \cdot \frac{\pi}{2} = -\pi$ **(4p)**

$\Longrightarrow f$ has a jump discontinuity at $x_2 = -2$ **(1p)**

3. (9 points) Find the equation of the tangent line to the function $f(x) = \frac{e^{x^2} \ln(2x+5)}{\cos x}$ at $x_0 = 0$.

Solution. $f'(x) = \frac{1}{\cos^2 x} \left(\left(e^{x^2} \cdot 2x \cdot \ln(2x+5) + e^{x^2} \cdot \frac{1}{2x+5} \cdot 2 \right) \cdot \cos x - e^{x^2} \ln(2x+5) \cdot (-\sin x) \right)$ **(5p)**

$f(0) = \ln 5$, $f'(0) = \frac{2}{5}$ **(1p)**

The equation of the tangent line is $y = f(0) + f'(0)(x - 0)$, that is, $y = \ln 5 + \frac{2}{5} x$ **(3p)**

4. (9+9+9 points) Calculate the following limits:

a) $\lim\limits_{x \to 0} \frac{\sqrt{1+8x} - e^{4x}}{x \sin 2x}$ b) $\lim\limits_{x \to 0} (\cos x)^{\frac{1}{\sin^2 x}}$ c) $\lim\limits_{x \to \infty} \frac{\sinh(2x+3)}{\cosh(2x-5)}$

Solution. a) The limit has the form $\frac{0}{0}$ $\Longrightarrow$ L'Hospital's rule can be applied:

$$\lim_{x \to 0} \frac{\sqrt{1+8x} - e^{4x}}{x \sin 2x} \overset{L'H}{=} \lim_{x \to 0} \frac{\frac{1}{2}(1+8x)^{-\frac{1}{2}} \cdot 8 - 4e^{4x}}{\sin 2x + 2x \cos 2x} \quad \textbf{(4}\,\boldsymbol{p}\textbf{)}$$

$$\overset{L'H}{=} \lim_{x\to 0} \frac{-\frac{1}{4}(1+8x)^{-\frac{3}{2}}\cdot 64 - 16e^{4x}}{2\cos 2x + 2\cos 2x + 2x(-\sin 2x)\cdot 2} \ \textbf{(3 p)} = \frac{-\frac{1}{4}\cdot 64 - 16}{2+2+0} = -8 \ \textbf{(2p)}$$

b) The limit has the form 1^∞: $(\cos x)^{\frac{1}{\sin^2 x}} = e^{\ln\left((\cos x)^{\frac{1}{\sin^2 x}}\right)} = e^{\left(\frac{1}{\sin^2 x}\ln(\cos x)\right)}$ **(3p)**
The limit of the power has the form $\frac{0}{0}$:

$$\lim_{x\to 0}\frac{\ln(\cos x)}{\sin^2 x} \overset{L'H}{=} \lim_{x\to 0}\frac{\frac{1}{\cos x}(-\sin x)}{2\sin x\cos x} = \lim_{x\to 0}\frac{1}{2\cos^2 x} = \frac{1}{2} \ \textbf{(4p)}$$

$$\Longrightarrow \lim_{x\to 0}(\cos x)^{\frac{1}{\sin^2 x}} = e^{\frac{1}{2}} = \sqrt{e} \ \textbf{(2p)}$$

c) By the definition of the functions:

$$\lim_{x\to\infty}\frac{\sinh(2x+3)}{\cosh(2x-5)} = \lim_{x\to\infty}\frac{e^{2x+3}-e^{-(2x+3)}}{e^{2x-5}+e^{-(2x-5)}} \ \textbf{(3p)} = \lim_{x\to\infty}\frac{e^{2x}}{e^{2x}}\,\frac{e^3-e^{-4x-3}}{e^{-5}+e^{-4x+5}} \ \textbf{(3p)} = \frac{e^3-0}{e^{-5}+0} = e^8 \ \textbf{(3p)}$$

5. (9 points) Find the values of the parameters such that the following function be differentiable on $\mathbb{R}$: $f(x) = \begin{cases} \frac{a}{x^2+1} & \text{if } x \geq 1 \\ b x^4 + 1 & \text{if } x < 1 \end{cases}$

Solution. The function is differentiable for all a, b except $x = 1$.

If f is continuous at $x = 1$ then $\lim_{x\to 1+0} f(x) = \lim_{x\to 1-0} f(x) = f(1) \Longrightarrow \frac{a}{2} = b + 1$ **(3p)**

$$f'(x) = \begin{cases} -\frac{a}{(x^2+1)^2}\cdot 2x & \text{if } x > 1 \\ 4bx^3 & \text{if } x < 1 \end{cases} \quad \textbf{(2p)}$$

If f is differentiable at $x = 1$ then $\lim_{x\to 1+0} f'(x) = \lim_{x\to 1-0} f'(x) \Longrightarrow -\frac{a}{2} = 4b$ **(3p)**.

The solution of the equation system is $a = \frac{8}{5}$, $b = -\frac{1}{5}$. **(1p)**

6. (18 points) Analyze the following function and plot its graph: $f(x) = \frac{x^2+x+1}{x^2+1}$.

Solution.

$D_f = \mathbb{R}$; $f(x) \neq 0$; $\lim_{x\to\pm\infty} f(x) = 1$

$$f'(x) = \frac{1-x^2}{(x^2+1)^2} = 0 \iff x = \pm 1 \ \textbf{(3p)}$$

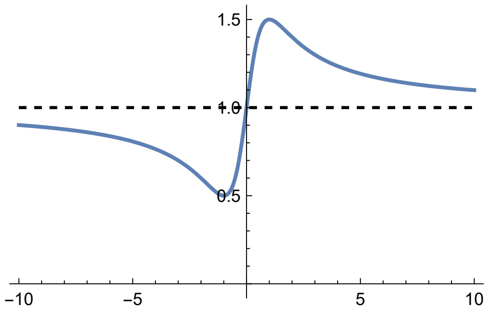
x	x<-1	x=-1	-1<x<1	x=1	x>1
f'	-	0	+	0	-
f	↘	min: $\frac{1}{2}$	↗	max: $\frac{3}{2}$	↘

(4p)

$$f''(x) = \frac{2x(x^2-3)}{(1+x^2)^3} = 0 \iff x = 0 \text{ or } x = \pm\sqrt{3} \ \textbf{(3p)}$$

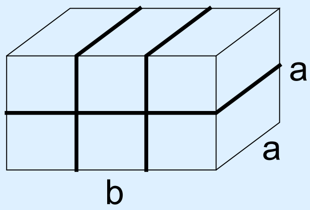
x	$x<-\sqrt{3}$	$x=-\sqrt{3}$	$-\sqrt{3}<x<0$	x=0	$0<x<\sqrt{3}$	$x=\sqrt{3}$	$x>\sqrt{3}$
f''	–	0	+	0	–	0	+
f	$\cap$	infl: $\frac{4-\sqrt{3}}{4}$	$\cup$	infl: 1	$\cap$	infl: $\frac{4+\sqrt{3}}{4}$	$\cup$

(4p)



$\Longrightarrow R_f = \left[\frac{1}{2}, \frac{3}{2}\right]$ **(4p)**

7. (9 points) The volume of a rectangular box with a square base is $V = 20\,\text{dm}^3$. A rope is tied around the box as shown in the figure. Find the dimensions of the box if the length of the rope is minimal.



Solution. The volume of the box is $V = a^2\, b = 20\ (\text{dm}^3)$, the length of the rope is $L = 2\,b + 10\,a$ **(1p)**.

Expressing b from V and substituting into L, we get that $L(a) = \frac{40}{a^2} + 10\,a$. **(2p)** We want to find the minimum of L on the interval $a \in (0, \infty)$. **(1p)**

$L'(a) = -\frac{80}{a^3} + 10\,a$ **(1p)**.

$L'(a) = 0 \Longrightarrow a = 2,\ b = 5$ **(2p)**.

$L''(a) = \frac{240}{a^4} \Longrightarrow L''(2) > 0$ (or, $L'(a) < 0$ if $a < 2$ and $L'(a) > 0$ if $a > 2$)

$\Longrightarrow L$ ha a local minimum at $a = 2$ **(1p)**, which is a global minimum on the interval $(0, \infty)$ **(1p)**.

8. (9 points) Estimate the value of $\sqrt[3]{1.2}$ by the Taylor polynomial of order 2 of $f(x) = \sqrt[3]{1+x}$ at center 0. Give an upper bound for the error of the approximation.

Solution.

$f(x) = \sqrt[3]{1+x}$ $\qquad f(0) = 1$

$f'(x) = \frac{1}{3\,(1+x)^{2/3}}$ $\qquad f'(0) = \frac{1}{3}$

$f''(x) = -\frac{2}{9\,(1+x)^{5/3}}$ $\qquad f''(0) = -\frac{2}{9}$

$f'''(x) = \frac{10}{27\,(1+x)^{8/3}}$ **(3p)**

The Taylor polynomial of order 2:

$$T_2(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 = 1 + \frac{1}{3}x - \frac{2}{9\cdot 2!}x^2$$

If $x = 0.2$ then $\sqrt[3]{1.2} \approx T_2(0.2) = 1 + \frac{1}{2}\cdot 0.2 - \frac{2}{9\cdot 2!}0.2^2 \approx 1.06222$ **(2p)**

Lagrange remainder term: $R_2(x) = \frac{f^{(3)}(\xi)}{3!}(x-x_0)^3$, where $x_0 = 0$, $x = 0.2$, $0 < \xi < 0.2$

The error estimation:

$$|E| = |R_2(x)| = \left|\frac{10}{27(1+\xi)^{8/3}}\cdot\frac{1}{3!}(0.2-0)^3\right| =$$

$$= \frac{10}{27(1+\xi)^{8/2}}\frac{1}{3!}0.2^3 < \frac{10}{27(1+0)^{5/2}}\frac{1}{3!}0.2^3 \approx 0.000493827$$ **(4p)**

9.* (10 points - BONUS) Without the help of a calculator, by investigating the function $f(x) = x - e\ln(x)$, show that $e^3 > 3^e$.

Solution. $f'(x) = 1 - \frac{e}{x}$

$f(e) = 0$

$f'(x) = 1 - \frac{e}{x} = \frac{x-e}{x} > 0$, if $x > e \Longrightarrow \forall x > e\colon\ f(x) > 0$

$\Longrightarrow$ Lagrange's mean value theorem can by applied on $[e, x]$:

$\frac{f(x)-f(e)}{x-e} = \frac{f(x)}{x-e} = f'(c) > 0.$

$3 > e \Longrightarrow f(3) = 3 - e\ln 3 > 0 \Longrightarrow 3 > e\ln 3 = \ln 3^e \Longrightarrow e^3 > 3^e$ **(10p)**

Or: $f'(x) = 1 - \frac{e}{x} = \frac{x-e}{x} > 0$, if $x > e \Longrightarrow f$ is strictly monotonically increasing on (e, ∞).

$3 > e \Longrightarrow f(3) > f(e) = 0 \Longrightarrow \ldots$

Remark: $e^3 \approx 20.0855$, $3^e \approx 19.813$