22nd and 23rd lectures

Definite integral

The Riemann integral

Definition. A partition of an interval $[a, b]$ is a finite set $P = \{x_0, x_1, \ldots x_n\}$ such that

$$a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b.$$

Definition. Assume that $f : [a, b] \longrightarrow \mathbb{R}$ is bounded and $P = \{x_0, x_1, \ldots x_n\}$ is a partition of $[a, b]$. Let

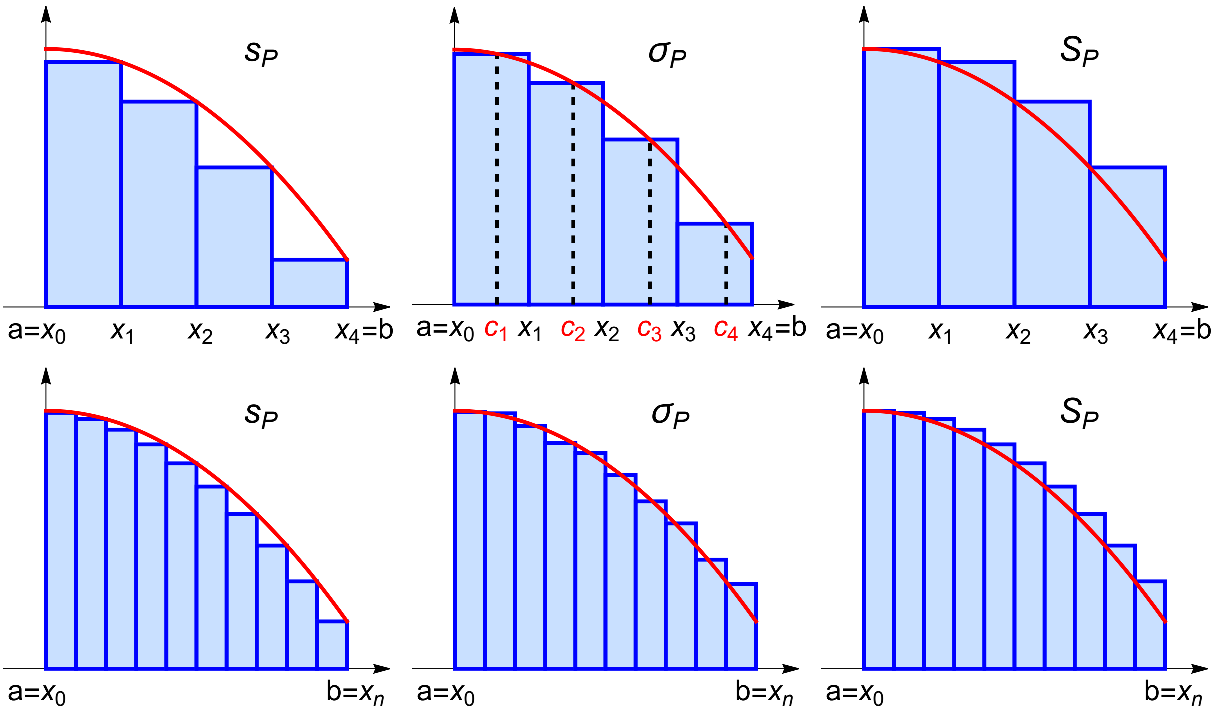
$$m_k := \inf\{f(x) : x \in [x_{k-1}, x_k]\}$$
$$M_k := \sup\{f(x) : x \in [x_{k-1}, x_k]\}$$

The **lower Darboux sum** of f with respect to P is $s_P = \sum_{k=1}^{n} m_k (x_k - x_{k-1})$.

The **upper Darboux sum** of f with respect to P is $S_P = \sum_{k=1}^{n} M_k (x_k - x_{k-1})$.

The **Riemann sum** of f with respect to P is $\sigma_P = \sum_{k=1}^{n} f(c_k)\,(x_k - x_{k-1})$, where

$c_k \in [x_{k-1}, x_k]$ is arbitrary. The points c_k are called the **evaluation points**.



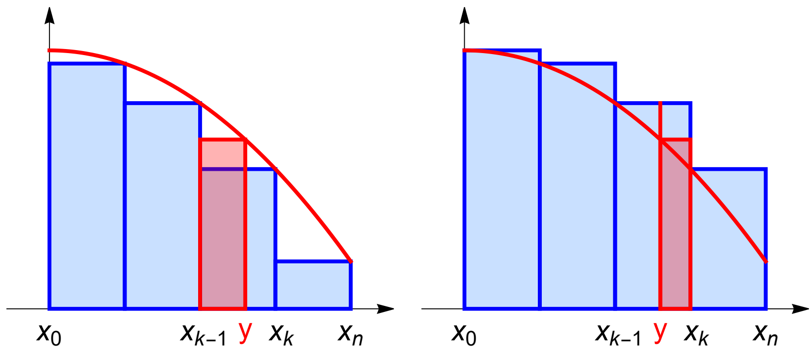
Statement. $s_P \le \sigma_P \le S_P$ for all partitions P.

Proof. It follows from the fact that $m_k \le f(c_k) \le M_k$ on each subinterval $[x_{k-1}, x_k]$.

Definition. Let P_1 and P_2 be partitions of $[a, b]$. If P_2 contains all points of P_1 and some additional points then P_2 is a refinement of P_1.

Theorem. If P_2 is a refinement of P_1 then $s_{P_1} \leq s_{P_2}$ and $S_{P_1} \leq S_{P_2}$,
that is, by refining a partition, the lower Darboux sum cannot decrease and the upper Darboux sum cannot increase.

Proof. Let P_2 be the partition that is obtained from $P_1 = \{x_0, x_1, \ldots, x_n\}$ by adding the point $x_{k-1} < y < x_k$. We prove $s_{P_1} \leq s_{P_2}$.
Let $A = \inf\{f(x) : x \in [x_{k-1}, y]\}$ and $B = \inf\{f(x) : x \in [y, x_k]\}$.
Then $m_k(x_k - x_{k-1}) = m_k(y - x_{k-1}) + m_k(x_k - y) \leq A(y - x_{k-1}) + B(x_k - y)$
$\Longrightarrow s_{P_2} - s_{P_1} = A(y - x_{k-1}) + B(x_k - y) - m_k(x_k - x_{k-1}) \geq 0.$



Theorem. $s_{P_1} \leq S_{P_2}$ for any partitions P_1 and P_2 of $[a, b]$, that is,
any lower Darboux sum is less than or equal to any upper Darboux sum.

Proof. Let $P_3 = P_1 \cup P_2 \Longrightarrow P_3$ is a refinement of P_1 and $P_2 \Longrightarrow s_{P_1} \leq s_{P_3} \leq S_{P_3} \leq S_{P_2}$

Definition. Assume that $f : [a, b] \longrightarrow \mathbb{R}$ is bounded.

The **lower Darboux integral** of f is $\underline{\int_a^b} f = \sup\{s_P : P \text{ is a partition of } [a, b]\}$.

The **upper Darboux integral** of f is $\overline{\int_a^b} f = \inf\{S_P : P \text{ is a partition of } [a, b]\}$.

Consequence: $\underline{\int_a^b} f \leq \overline{\int_a^b} f$

Definition. If $f : [a, b] \longrightarrow \mathbb{R}$ is bounded and $I = \underline{\int_a^b} f = \overline{\int_a^b} f$ then f is **Riemann integrable** on $[a, b]$.

In this case the Riemann integral of f on $[a, b]$ is denoted as
$I = \int_a^b f(x)\,dx$ or $I = \int_a^b f$. (f is called the integrand.)

Notation. $R[a, b]$ denotes the set of those functions that are Riemann integrable on $[a, b]$

Remark. If $f : [a, b] \longrightarrow \mathbb{R}$ is not bounded on $[a, b]$ or bounded but $\underline{\int_a^b} f < \overline{\int_a^b} f$ then f is not Riemann integrable on $[a, b]$.

Example: Let $f(x) = c \in \mathbb{R}$, $\int_a^b c\,dx = ?$

$$s_P = \sum_{k=1}^{n} m_k(x_k - x_{k-1}) = \sum_{k=1}^{n} c(x_k - x_{k-1}) = c(b - a),$$

$$S_P = \sum_{k=1}^{n} M_k (x_k - x_{k-1}) = \sum_{k=1}^{n} c(x_k - x_{k-1}) = c(b-a) \text{ for all partitions } P.$$

$$\underline{\int_a^b} f = \sup\{s_P\} = c(b-a) = \inf\{S_P\} = \overline{\int_a^b} f \implies \int_a^b c\, dx = c(b-a)$$

Example: The Dirichlet function $f(x) = \begin{cases} 1 & \text{if } x \in [0,1] \cap \mathbb{Q} \\ 0 & \text{if } x \in [0,1] \setminus \mathbb{Q} \end{cases}$ is bounded, and for all partitions P of $[0, 1]$, $s_P = 0$ and $S_P = 1$

$\implies \underline{\int_a^b} f = 0$ and $\overline{\int_a^b} f = 1$

$\implies f$ is not integrable on $[0, 1]$.

Necessary and sufficient conditions for Riemann integrability

Definition. The **mesh** or **norm of a partition** is the maximal distance between adjacent points in the partition: $\Delta P = \max_{k \in \{1,\ldots,n\}} (x_k - x_{k-1})$.

Statement. Assume that $f : [a, b] \longrightarrow \mathbb{R}$ is bounded and (P_n) is a sequence of partitions of $[a, b]$.

If $\lim_{n\to\infty} \Delta P_n = 0$ then $\lim_{n\to\infty} s_{P_n} = \underline{\int_a^b} f$ and $\lim_{n\to\infty} S_{P_n} = \overline{\int_a^b} f$

Statement. a) If $\exists \int_a^b f(x)\, dx \implies$ for all partition sequences (P_n) for which $\lim_{n\to\infty} \Delta P_n = 0$:

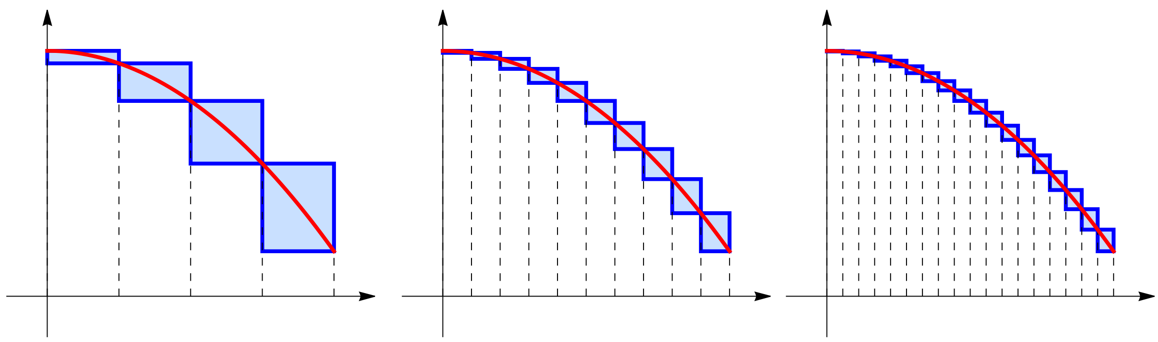
$$\lim_{n\to\infty} s_{P_n} = \lim_{n\to\infty} S_{P_n} = \int_a^b f(x)\, dx.$$

b) If (P_n) is a partition sequence for which $\lim_{n\to\infty} \Delta P_n = 0$ and $\lim_{n\to\infty} s_{P_n} = \lim_{n\to\infty} S_{P_n} = I$

$$\implies \exists \int_a^b f(x)\, dx = I.$$

Definition. Assume that $f : [a, b] \longrightarrow \mathbb{R}$ is bounded and $P = \{x_0, x_1, \ldots x_n\}$ is a partition of $[a, b]$. Then the oscillation sum of f related to the partition P is

$$O_P = \sum_{k=1}^{n} (M_k - m_k)(x_k - x_{k-1}) = S_P - s_P.$$



Theorem (Riemann's criterion for integrability). Assume that $f : [a, b] \longrightarrow \mathbb{R}$ is bounded.
f is integrable on $[a, b] \iff$ for all $\varepsilon > 0$ there exists a partition P such that $O_P = S_P - s_P < \varepsilon$.

Proof. $\implies$: Assume that f is integrable and $\varepsilon > 0$. Then there exist partitions P_1 and P_2 such that

$$0 \le S_{P_2} - \overline{\int_a^b} f < \frac{\varepsilon}{2} \text{ and } 0 \le \underline{\int_a^b} f - s_{P_1} < \frac{\varepsilon}{2}.$$

Let $P = P_1 \cup P_2$ (P is a common refinement of P_1 and P_2). Then $s_{P_1} \le s_P \le S_P \le S_{P_2}$, so

$$0 \le O_P = S_P - s_P \le S_{P_2} - s_{P_1} = \left(S_{P_2} - \overline{\int_a^b}\right) + \left(\underline{\int_a^b} f - s_{P_1}\right) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$\Longleftarrow$: For any partition P, $s_P \le \underline{\int_a^b} f \le \overline{\int_a^b} f \le S_P$, so

$$0 \le \overline{\int_a^b} f - \underline{\int_a^b} f \le S_P - s_P = O_P < \varepsilon \text{ for all } \varepsilon > 0 \Longrightarrow \overline{\int_a^b} f = \underline{\int_a^b} f,$$ that is, f is integrable.

Remark. Recall that the **Riemann sum** of f with respect to the partition P is

$\sigma_P = \sum_{k=1}^{n} f(c_k)\,(x_k - x_{k-1})$, where the evaluation points $c_k \in [x_{k-1}, x_k]$ are arbitrary and

$s_P \le \sigma_P \le S_P$ for all partitions P.

Theorem. Assume that $f : [a, b] \longrightarrow \mathbb{R}$ is bounded. Then

1. $\exists \int_a^b f(x)\,dx = I \Longrightarrow$ for all partition sequences (P_n) for which $\lim_{n\to\infty} \Delta P_n = 0$:

$\lim_{n\to\infty} \sigma_{P_n} = \int_a^b f(x)\,dx = I$ (independent of the choice of the evaluation points).

2. $\exists \int_a^b f(x)\,dx = I \Longleftarrow$ there exists a partition sequence (P_n) for which $\lim_{n\to\infty} \Delta P_n = 0$

and $\exists \lim_{n\to\infty} \sigma_{P_n} = I$ (independent of the choice of the evaluation points).

Remark. The proof of part 1. is obvious, since $s_{P_n} \le \sigma_{P_n} \le S_{P_n}$ and $\lim_{n\to\infty} s_{P_n} = \lim_{n\to\infty} S_{P_n} = I$.

Remark. It is important that the limit exists independent of the choice of $c_k \in [x_{k-1}, x_k]$ in the Riemann sum. For example, assume that f is the Dirichlet function on $[a, b]$ and (P_n) is a sequence of partitions for which $\lim_{n\to\infty} \Delta P_n = 0$.

If c_k is rational: $\sigma_{P_n} = \sum_{k=1}^{n} 1 \cdot (x_k - x_{k-1}) = 1 \cdot (b - a) \longrightarrow b - a$

If c_k is irrational: $\sigma_{P_n} = \sum_{k=1}^{n} 0 \cdot (x_k - x_{k-1}) = 0 \longrightarrow 0$

$\Longrightarrow$ the Dirichlet function is not integrable on any interval.

Sufficient conditions for Riemann integrability

Theorem. If f is monotonic and bounded on $[a, b]$ then f is Riemann integrable on $[a, b]$.

Proof. Assume that f is **monotonically increasing**.

1) If $f(a) = f(b)$ then f is constant, so $f \in R[a, b]$.
2) If $f(a) < f(b)$ then we show that for all $\varepsilon > 0$ there exists a partition P such that the oscillation sum $O_P = S_P - s_P < \varepsilon$.
3) Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition with mesh

$$\Delta P = \max_{k \in \{1,\ldots,n\}} (x_k - x_{k-1}) < \delta = \frac{\varepsilon}{f(b) - f(a)} > 0.$$

4) Then for the oscillation sum we get that

$$O_P = S_P - s_P = \sum_{k=1}^{n} (M_k - m_k)(x_k - x_{k-1}) = \sum_{k=1}^{n} (f(x_k) - f(x_{k-1}))(x_k - x_{k-1}) <$$

$$< \delta \sum_{k=1}^{n} (f(x_k) - f(x_{k-1})) = \delta(f(b) - f(a)) = \varepsilon.$$

Theorem. If $f : [a, b] \longrightarrow \mathbb{R}$ is continuous then f is Riemann integrable on $[a, b]$.

Proof. 1) We prove that for all $\varepsilon > 0$ there exists a partition P such that the oscillation sum $O_P = S_P - s_P < \varepsilon$.

2) f is continuous on $[a, b]$ $\Longrightarrow$ f is bounded and also uniformly continuous on $[a, b]$.

$\Longrightarrow$ for $\frac{\varepsilon}{b-a} > 0$ there exists $\delta > 0$ such that $\forall\, x, y \in [a, b]$,

$$|x - y| < \delta \Longrightarrow |f(x) - f(y)| < \frac{\varepsilon}{b-a}.$$

3) Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition with mesh $\Delta P = \max\limits_{k \in \{1,\ldots,n\}} (x_k - x_{k-1}) < \delta$.

4) f is continuous on $[x_{k-1}, x_k]$ $\Longrightarrow$ by the extreme value theorem f has a minimum for some $c_k \in [x_{k-1}, x_k]$ and a maximum for some $d_k \in [x_{k-1}, x_k]$, let $f(c_k) = m_k$, $f(d_k) = M_k$.

5) Then obviously $|d_k - c_k| < \delta$, so for the oscillation sum we get that

$$O_P = S_P - s_P = \sum_{k=1}^{n} (M_k - m_k)(x_k - x_{k-1}) = \sum_{k=1}^{n} (f(d_k) - f(c_k))(x_k - x_{k-1}) =$$

$$= \sum_{k=1}^{n} |f(d_k) - f(c_k)| (x_k - x_{k-1}) < \sum_{k=1}^{n} \frac{\varepsilon}{b-a} (x_k - x_{k-1}) =$$

$$= \frac{\varepsilon}{b-a} \sum_{k=1}^{n} (x_k - x_{k-1}) = \frac{\varepsilon}{b-a} (b-a) = \varepsilon.$$

Theorem. If $f : [a, b] \longrightarrow \mathbb{R}$ is bounded and continuous except finitely many points then f is Riemann integrable on $[a, b]$.

Proof. 1) We prove it in the case of one point. Let $c \in [a, b]$ and assume that f is continuous on $[a, b] \setminus \{c\}$. Let $K > 0$ be such that $|f(x)| \leq K$ for all $x \in [a, b]$. We show that for all $\varepsilon > 0$ there exists a partition P such that $O_P < \varepsilon$.

2) If $c - \frac{\varepsilon}{8K} > a$ then let $c_1 = c - \frac{\varepsilon}{8K}$ and let P_1 be a partition of $[a, c_1]$ such that $O_{P_1} < \frac{\varepsilon}{4}$.
Such a partition exists since f is continuous on $[a, c_1]$.
If $c - \frac{\varepsilon}{8K} \leq a$ then let $c_1 = a$ and $P_1 = \{a\}$.

3) If $c + \frac{\varepsilon}{8K} < b$ then let $c_2 = c + \frac{\varepsilon}{8K}$ and let P_2 be a partition of $[c_2, b]$ such that $O_{P_2} < \frac{\varepsilon}{4}$.
Such a partition exists since f is continuous on $[c_2, b]$.
If $c + \frac{\varepsilon}{8K} \geq b$ then let $c_2 = b$ and $P_2 = \{b\}$.

4) Then $P = P_1 \cup P_2$ is a suitable choice.

Remark. If $f, g : [a, b] \longrightarrow \mathbb{R}$, f is Riemann integrable and $f(x) = g(x)$ except finitely many points in $[a, b]$ then g is Riemann integrable and $\int_a^b f = \int_a^b g$.

Newton-Leibniz formula

Theorem (First fundamental theorem of calculus, Newton-Leibniz formula).
If $f : [a, b] \longrightarrow \mathbb{R}$ is Riemann integrable and $F : [a, b] \longrightarrow \mathbb{R}$ is an antiderivative of f, that is, $F'(x) = f(x)$ for all $x \in [a, b]$, then

$$\int_a^b f(x)\,\mathrm{dx} = F(b) - F(a) = [F(x)]_a^b$$

Proof. Let (P_n) be a partition sequence of $[a, b]$ such that $\lim_{n\to\infty} \Delta P_n = 0$.

For all $k \in \{1, 2, \ldots, n\}$, F is continuous on $[x_{k-1}, x_k]$ and differentiable on (x_{k-1}, x_k), so by Lagrange's mean value theorem there exists $x_{k-1} < c_k < x_k$ such that

$$\frac{F(x_k) - F(x_{k-1})}{x_k - x_{k-1}} = F'(c_k) = f(c_k) \implies F(x_k) - F(x_{k-1}) = f(c_k)(x_k - x_{k-1})$$

$$\implies F(b) - F(a) = (F(x_1) - F(x_0)) + (F(x_2) - F(x_1)) + \ldots + (F(x_n) - F(x_{n-1})) =$$

$$= \sum_{k=1}^{n} (F(x_k) - F(x_{k-1})) = \sum_{k=1}^{n} f(c_k)(x_k - x_{k-1}) = \sigma_{P_n}$$

$$\implies F(b) - F(a) = \sigma_{P_n}$$

Taking the limits of both sides: $\lim_{n\to\infty} (F(b) - F(a)) = \lim_{n\to\infty} \sigma_{P_n}$

The left-hand side is independent of n and since f is integrable then the limit of the right-hand side is the integral of f, so

$$F(b) - F(a) = \int_a^b f(x)\,\mathrm{dx}.$$

Remark. The geometrical meaning of $\int_a^b f$ is the signed area under the graph of f on $[a, b]$.

Remark. Both conditions of the theorem are important as the following examples show.

Examples

Example 1. Let $F(x) = \begin{cases} x^2 \sin \frac{1}{x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$, then $F'(x) = f(x) = \begin{cases} 2x \sin \frac{1}{x^2} - \frac{2}{x} \cos \frac{1}{x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$.

f has an antiderivative, however, $\int_0^1 f(x)\,\mathrm{dx}$ doesn't exist, since f is not bounded.

Example 1. $\int_0^5 \operatorname{sign}(x^2 - 5x + 6)\,\mathrm{dx}$ exists, since f is continuous except 2 points. However, by Darboux's theorem, f doesn't have an antiderivative, since f has jump discontinuities.

Properties of Riemann integrable functions

Definition. If $f \in R[a, b]$ $\int_b^a f(x)\,\mathrm{dx} := -\int_a^b f(x)\,\mathrm{dx}$, $\int_a^a f(x)\,\mathrm{dx} := 0$

Theorem. Let $f, g \in R[a, b]$ and $\lambda \in \mathbb{R}$. Then

(1) $\lambda f,\ f + g,\ f - g \in R[a, b]$ and $\int_a^b \lambda f = \lambda \int_a^b f$, $\int_a^b (f \pm g) = \int_a^b f \pm \int_a^b g$

(2) $[\alpha, \beta] \subset [a, b] \implies f \in R[\alpha, \beta]$

(3) $a < c < b \implies \int_a^b f = \int_a^c f + \int_c^b f$

(4) $f(x) \le g(x)\ \forall x \in [a, b] \implies \int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$

(5) $|f| \in R[a, b] \implies \left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$

(6) $\inf_{[a,b]} f \le \frac{1}{b-a}\int_a^b f \le \sup_{[a,b]} f$

Integration by parts

Theorem. If f and g are continuously differentiable on $[a, b]$ then $\int_a^b f' g = [f g]_a^b - \int_a^b f g'$

Integration by substitution

Theorem. If g is continuously differentiable, strictly monotonic, $[a, b] \subset D_g$ and

f is continuous on $[a, b]$ then $\int_a^b f(x)\,dx = \int_{g^{-1}(a)}^{g^{-1}(b)} f(g(t))\,g'(t)\,dt.$

Example. $I = \int_0^{\ln 2} \sqrt{e^x - 1}\,dx = ?$

Solution. Substitution: $t = \sqrt{e^x - 1} \implies x = x(t) = \ln(t^2 + 1)$

$$x'(t) = \frac{dx}{dt} = \frac{1}{t^2+1} \cdot 2t \implies dx = \frac{2t}{t^2+1}\,dt$$

The bounds will change: $x_1 = 0 \implies t_1 = \sqrt{e^0 - 1} = 0$

$$x_2 = \ln 2 \implies t_2 = \sqrt{e^{\ln 2} - 1} = \sqrt{2-1} = 1$$

$$I = \int_0^{\ln 2} \sqrt{e^x - 1}\,dx = \int_{t_1}^{t_2} t \cdot \frac{2t}{t^2+1}\,dt = \int_0^1 \frac{2t^2}{t^2+1}\,dt = \int_0^1 \frac{2(t^2+1) - 2}{t^2+1}\,dt = \int_0^1 \left(2 - \frac{2}{t^2+1}\right) dt =$$

$$= [2t - 2\,\text{arctg}\,t]_0^1 = (2 \cdot 1 - 2\,\text{arctg}1) - (0 - 0) = 2 - \frac{\pi}{2}$$

Lebesgue’s theorem

Definition. We say that the set $A \subset \mathbb{R}$ has **Lebesgue measure 0** if for all $\varepsilon > 0$ there exist

sequences (x_n) and (y_n) such that $x_n \le y_n$, $A \subset \bigcup_{n=1}^{\infty} [x_n, y_n]$ and $\sum_{n=1}^{\infty} (y_n - x_n) < \varepsilon$.

(That is, A can be covered with countably many intervals such that their total length is less than ε.)

Examples. 1) Any countable set of $\mathbb{R}$ has Lebesgue measure 0, for example $\mathbb{N}$, $\mathbb{Z}$ or $\mathbb{Q}$.

2) The Cantor set is defined in the following way. Let $C_0 = [0, 1]$.

C_1 is obtained from C_0 by deleting the open middle third from C_0, that is,

$C_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$.

C_2 is obtained from C_1 by deleting the open middle thirds from C_1, that is,

$C_2 = \left[0, \frac{1}{9}\right] \bigcup \left[\frac{2}{9}, \frac{1}{3}\right] \bigcup \left[\frac{2}{3}, \frac{7}{9}\right] \bigcup \left[\frac{8}{9}, 1\right]$

Continuing this process, C_{n+1} is obtained from C_n by deleting the open middle thirds of each of these intervals. The Cantor set is $C = \bigcap_{n \in \mathbb{N}} C_n$.

It can proved that the Cantor set is uncountable but has Lebesgue measure 0.

Theorem (Lebesgue). The function $f : [a, b] \longrightarrow \mathbb{R}$ is Riemann integrable if and only if it is bounded and the set of discontinuities of f has Lebesgue measure 0.

Remark. If $f : [a, b] \longrightarrow \mathbb{R}$ is monotonic then f has at most countably many discontinuities (and they are jump discontinuities), so by Lebesgue's theorem f is Riemann integrable.

Example*. The Riemann function is defined as

$$f : \mathbb{R} \longrightarrow \mathbb{R},\ f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \\ \frac{1}{q} & \text{if } x = \frac{p}{q} \text{ where } p \in \mathbb{Z}, \text{ and } q \in \mathbb{N}^+ \text{ are coprimes} \end{cases}$$

Prove that

a) $\lim_{x \to a} f(x) = 0 \ \forall a \in \mathbb{R}$;

a) f is continuous at all irrational numbers;

b) f is discontinuous at all rational numbers.

Solution. If $q \in \mathbb{N}^+$ is fixed then the set $\mathbb{Z} \cdot \frac{1}{q} = \left\{\frac{k}{q} : k \in \mathbb{Z}\right\}$ does not have any real limit points.

Therefore a finite union of such sets, $A_n = \left\{\frac{p}{q} : p \in \mathbb{Z},\ q \in \{1, 2, \ldots, n\}\right\}$ does not have any limit points either. If $x \in \mathbb{R} \setminus A_n$ the $\left| f(x) \right| < \frac{1}{n}$, so for all $x_0 \in \mathbb{R}$, $\lim_{x \to x_0} f(x) = 0$.

$\Longrightarrow$ f is continuous at all irrational points and has a removable discontinuity at all rational points.

The Riemann function is bounded and the set of discontinuities is countable, so it has Lebesgue measure 0 $\Longrightarrow$ f is Riemann integrable and $\int_a^b f(x)\, dx = 0$.

The integral function

Definition. Assume that f is Riemann integrable on $[a, b]$. Then the function

$$F(x) = \int_a^x f(t)\, dt,\ x \in [a, b]$$

is called the **integral function** of f.

Theorem (Second fundamental theorem of calculus).

Assume that f is Riemann integrable on $[a, b]$ and $F(x) = \int_a^x f(t)\,dt,\ x \in [a, b]$. Then

1. F is Lipschitz continuous on $[a, b]$.
2. If f is continuous at $x_0 \in [a, b]$ then F is differentiable at x_0 and $F'(x_0) = f(x_0)$.

Proof. 1. Let $K = \sup_{[a,b]} |f(x)|$. If $K = 0$ then $f = 0$ so $F = 0$ is Lipschitz continuous.

If $K \neq 0$ then $0 < K \in \mathbb{R}$. Let $\varepsilon > 0$ and $\delta(\varepsilon) = \frac{\varepsilon}{K}$. If $x, y \in [a, b]$ such that $|x - y| < \delta$ then

$$|F(x) - F(y)| = \left|\int_a^x f(t)\,dt - \int_a^y f(t)\,dt\right| = \left|\int_y^x f(t)\,dt\right| \leq \left|\int_y^x |f(t)|\,dt\right| \leq \left|\int_y^x K\,dt\right| \leq$$

$$\leq K\,|x - y| < K\,\delta = \varepsilon \implies F \text{ is Lipschitz continuous.}$$

2. $F'(x_0) = \lim_{x \to x_0} \frac{F(x) - F(x_0)}{x - x_0} = f(x_0)$ if for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\left|\frac{F(x) - F(x_0)}{x - x_0} - f(x_0)\right| < \varepsilon \text{ if } 0 < |x - x_0| < \delta.$$

Let $\varepsilon > 0$. Since f is continuous at x_0 then $\exists\,\delta > 0$ such that $|f(x) - f(x_0)| < \varepsilon$ if $|x - x_0| < \delta$.
Then with this δ

$$\left|\frac{F(x) - F(x_0)}{x - x_0} - f(x_0)\right| = \left|\frac{F(x) - F(x_0) - f(x_0)(x - x_0)}{x - x_0}\right| = \left|\frac{\int_{x_0}^x f(t)\,dt - \int_{x_0}^x f(x_0)\,dt}{x - x_0}\right| =$$

$$= \left|\frac{\int_{x_0}^x (f(t) - f(x_0))\,dt}{x - x_0}\right| \leq \frac{\left|\int_{x_0}^x |f(t) - f(x_0)|\,dt\right|}{|x - x_0|} \leq \frac{\left|\int_{x_0}^x \varepsilon\,dt\right|}{|x - x_0|} = \frac{|\varepsilon(x - x_0)|}{|x - x_0|} = \varepsilon.$$

Consequence.

1. If f is continuous on $[a, b]$ and $F(x) = \int_a^x f(t)\,dt,\ x \in [a, b]$ then $F'(x) = f(x)\ \forall\,x \in [a, b]$.
2. Every continuous function has an antiderivative.

Examples

Example 1. Calculate the derivatives of the following functions:

a) $F(x) = \int_0^x \sin t^2\,dt,\ \ x \neq 0$ b) $G(x) = \int_0^{x^3} \sin t^2\,dt$ c) $H(x) = \int_{x^2}^{x^3} \sin t^2\,dt$

Solution. a) $F'(x) = \sin x^2$, since $f(t) = \sin(t^2)$ is continuous.

b) $G(x) = F(x^3) \implies G'(x) = F'(x^3)\cdot 3x^2 = \sin((x^3)^2)\cdot 3x^2 = \sin(x^6)\cdot 3x^2$

c) $H(x) = \int_0^{x^3} \sin t^2\,dt - \int_0^{x^2} \sin t^2\,dt = F(x^3) - F(x^2) \implies H'(x) = \sin(x^6)\cdot 3x^2 - \sin(x^4)\cdot 2x$

Example 2. $\lim_{x \to 0} \frac{\int_0^x \arctan t^2\,dt}{x^2} = ?$

Solution. The limit has the form $\frac{0}{0}$ and the numerator is differentiable since
$f(t) = \arctan t^2$ is continuous

$$\Longrightarrow \lim_{x\to 0}\frac{\int_0^x \arctan t^2\,\mathrm{dt}}{x^2} \overset{L'H}{=} \lim_{x\to 0}\frac{\arctan x^2}{2x} \overset{L'H}{=} \lim_{x\to 0}\frac{\frac{1}{1+x^4}\cdot 2x}{2} = 0$$

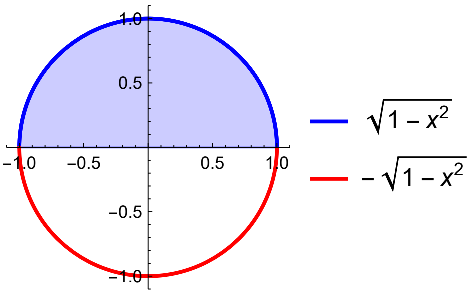
Applications

Area

Example. Calculate the area of the unit circle.

Solution. The equation of the circle with radius $r = 1$ centered at the origin is

$$x^2 + y^2 = 1 \Longrightarrow y^2 = 1 - x^2 \quad \Longrightarrow y = \pm\sqrt{1-x^2}$$



The area of the unit circle is $A = 2\int_{-1}^{1}\sqrt{1-x^2}\,\mathrm{dx}$

Substitution: $x = x(t) = \sin t \Longrightarrow t = \arcsin x$

$$x'(t) = \frac{\mathrm{dx}}{\mathrm{dt}} = \cos t \Longrightarrow \mathrm{dx} = \cos t\,\mathrm{dt}$$

The bounds will change: $x_1 = -1 \Longrightarrow t_1 = \arcsin(-1) = -\frac{\pi}{2}$

$$x_2 = 1 \Longrightarrow t_2 = \arcsin 1 = \frac{\pi}{2}$$

$$\Longrightarrow A = 2\int_{-1}^{1}\sqrt{1-x^2}\,\mathrm{dx} = \int_{-\pi/2}^{\pi/2} 2\sqrt{1-(\sin t)^2}\,\cos t\,\mathrm{dt} = 2\int_{-\pi/2}^{\pi/2}\cos t\cdot\cos t\,\mathrm{dt}$$

$$= \int_{-\pi/2}^{\pi/2} \mathbf{2\cos^2 t}\,\mathrm{dt} = \int_{-\pi/2}^{\pi/2}(\mathbf{1+\cos 2t})\,\mathrm{dt} = \left[t + \frac{\sin 2t}{2}\right]_{-\pi/2}^{\pi/2}$$

$$= \left(\frac{\pi}{2} + \frac{\sin\pi}{2}\right) - \left(-\frac{\pi}{2} + \frac{\sin(-\pi)}{2}\right) = \left(\frac{\pi}{2} + 0\right) - \left(-\frac{\pi}{2} + 0\right) = \pi$$

Arc length

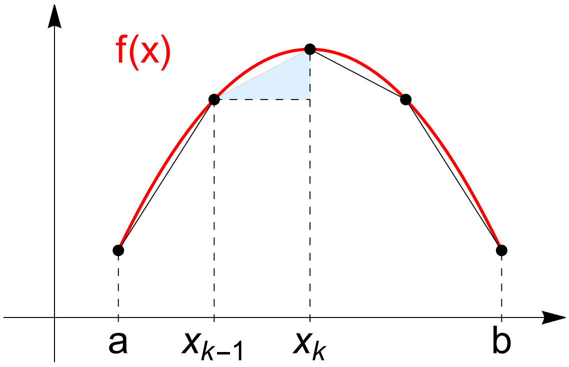
Theorem. Assume that $f : [a, b] \longrightarrow \mathbb{R}$ is continuously differentiable. Then the arc length of the graph of f is $L = \int_a^b \sqrt{1 + (f'(x))^2}\,\mathrm{dx}$.

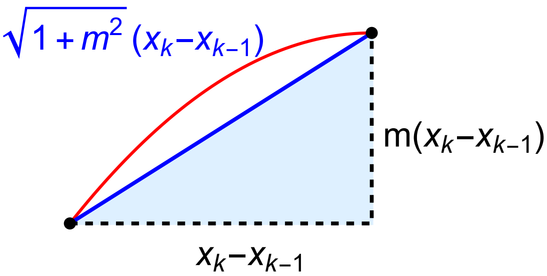
Remark. Let $a = x_0 < x_1 < x_2 < \ldots < x_n = b$ be a partition. If f is differentiable then by Lagrange's mean value theorem there exists $c_k \in (x_{k-1}, x_k)$ such that $m = f'(c_k)$, where m is the slope of the secant line connecting the points $(x_{k-1}, f(x_{k-1}))$ and $(x_k, f(x_k))$.

So the arc length can be approximated by the sum $\sum_{k=1}^{n}\sqrt{1+(f'(c_k))^2}\,(x_k - x_{k-1})$, which is the Riemann sum of the function $\sqrt{1+(f'(x))^2}$.

If f is continuously differentiable then the arc length of the graph of f is

$L = \int_a^b \sqrt{1+(f'(x))^2}\, dx.$





Example. Calculate the arc length of the unit circle.

Solution. Let $f(x) = \sqrt{1-x^2}$ if $x \in [-1, 1]$.

$$f'(x) = \frac{1}{2}(1-x^2)^{-\frac{1}{2}}(-2x) = -\frac{x}{\sqrt{1-x^2}}$$

$$\Longrightarrow \sqrt{1+(f'(x))^2} = \sqrt{1+\frac{x^2}{1-x^2}} = \sqrt{\frac{1}{1-x^2}} = \frac{1}{\sqrt{1-x^2}}$$

The arc length of the unit circle is

$$L = 2\int_{-1}^{1} \sqrt{1+(f'(x))^2}\, dx = 2\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\, dx = 2 \lim_{a\to -1+} \lim_{b\to 1-} \int_a^b \frac{1}{\sqrt{1-x^2}}\, dx =$$

$$= 2 \lim_{a\to -1+} \lim_{b\to 1-} [\arcsin x]_a^b = 2 \lim_{a\to -1+} \lim_{b\to 1-} (\arcsin b - \arcsin a) =$$

$$= 2(\arcsin 1 - \arcsin(-1)) = 2\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = 2\pi$$

Volume of solids of revolutions

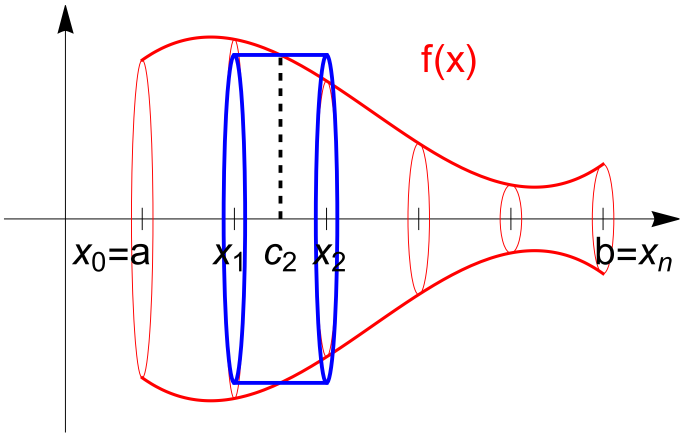
Theorem. Assume that $f : [a, b] \longrightarrow \mathbb{R}$ is continuous and nonnegative and the graph of f is rotated about the x axis. Then the volume of this solid of revolution is $V = \pi \int_a^b f^2(x)\, dx$.

Remark. If $a = x_0 < x_1 < x_2 < \ldots < x_n = b$ is a partition then the volume can be approximated by the sum $\sum_{k=1}^{n} (x_k - x_{k-1})\, \pi f^2(c_k)$ where $c_k \in [x_{k-1}, x_k]$ is arbitrary.

(Geometrically it means that the volume can be approximated by the sum of volumes of cylinders.)

This is the Riemann sum of the function $\pi f^2(x)$, so if f is continuous then the volume is

$V = \pi \int_a^b f^2(x)\, dx.$



Surface area of solids of revolutions

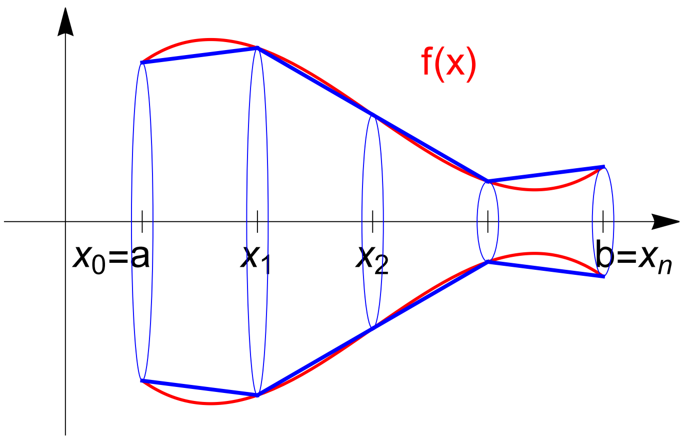
Theorem. Assume that $f : [a, b] \longrightarrow \mathbb{R}$ is continuously differentiable and nonnegative and the graph of f is rotated about the x axis. Then the surface area of this solid of revolution is $A = 2\pi \int_a^b f(x) \sqrt{1 + (f'(x))^2}\, dx$.

Remark. If $a = x_0 < x_1 < x_2 < \ldots < x_n = b$ is a partition then the surface area of the solid of revolution can be approximated by the sum

$$\sum_{k=1}^{n} \pi(f(x_{k-1}) + f(x_k)) \sqrt{1 + (f'(c_k))^2}\, (x_k - x_{k-1})$$

where $c_k \in [x_{k-1}, x_k]$ exists by the Lagrange intermediate value theorem if f is differentiable. (Geometrically it means that the surface area can be approximated by the sum of lateral surfaces of truncated cones.)

If f is continuously differentiable then $f(x_{k-1}) + f(x_k) \approx 2 f(c_k)$, so the above sum will be the Riemann sum of the function $2\pi f(x) \sqrt{1 + (f'(x))^2}$. Therefore if f is continuously differentiable then the surface area is $A = 2\pi \int_a^b f(x) \sqrt{1 + (f'(x))^2}\, dx$.



Exercise

Let $f(x) = \sqrt{r^2 - x^2}$, $-r \le x \le r$. Rotating the graph of f about the x axis, we get a sphere with radius r. Calculate the volume and surface area of the sphere.

Solution: 1. The volume can be calculated as $V = \pi \int_a^b f^2(x)\,\mathrm{dx}$

The integrand is $(f(x))^2 = r^2 - x^2$

The volume is $V = \pi \int_{-r}^{r} (r^2 - x^2)\,\mathrm{dx} = \pi \left[r^2 x - \frac{x^3}{3}\right]_{-r}^{r} =$

$= \pi\left(\left(r^3 - \frac{r^3}{3}\right) - \left(-r^3 + \frac{r^3}{3}\right)\right) = \frac{4 r^3 \pi}{3}$

2. The surface are can be calculated as $A = 2\pi \int_a^b f(x) \sqrt{1 + (f'(x))^2}\,\mathrm{dx}$

The derivative of f is $f'(x) = \left((r^2 - x^2)^{\frac{1}{2}}\right)' = \frac{1}{2} (r^2 - x^2)^{-\frac{1}{2}} \cdot (-2x) = -\frac{x}{\sqrt{r^2 - x^2}}$

$\Longrightarrow 1 + (f'(x))^2 = 1 + \frac{x^2}{r^2 - x^2} = \frac{r^2 - x^2 + x^2}{r^2 - x^2} = \frac{r^2}{r^2 - x^2}$

The integrand is $f(x) \sqrt{1 + (f'(x))^2} = \sqrt{r^2 - x^2} \cdot \sqrt{\frac{r^2}{r^2 - x^2}} = r$

The surface area is $A = 2\pi \int_{-r}^{r} r\,\mathrm{dx} = 2\pi \cdot [r\,x]_{-r}^{r} = 2\pi(r^2 - (-r^2)) = 4 r^2 \pi$

Additional exercises: Chapter 5, from page 86:
https://math.bme.hu/~tasnadi/merninf_anal_1/anal1_gyak.pdf