
21st lecture

Indefinite integrals

Antiderivative

Definition. If F is differentiable on the interval I and $F'(x) = f(x)$ for all $x \in I$
 F is an **antiderivative** or **primitive function** of f .

Example. If $f(x) = \sin x \cos x = \frac{1}{2} \sin 2x$ then

$F(x) = \frac{\sin^2 x}{2}$ and $G(x) = -\frac{\cos 2x}{4}$ are primitive functions of f on \mathbb{R}
since $F'(x) = G'(x) = f(x)$.

Theorem. If F and G are antiderivatives of f on the interval I then there exists
 $c \in \mathbb{R}$ such that $F(x) = G(x) + c$.

Proof. F and G are antiderivatives of $f \iff F' = G' \iff (F - G)' = 0 \iff$
 $\exists c \in \mathbb{R}: F - G = c \iff \exists c \in \mathbb{R}: F(x) = G(x) + c \quad \forall x \in I$.

Remark. This theorem holds only on an interval.

Definition. If f has an antiderivative then the set of antiderivatives of f is called
the **indefinite integral** of f :

$$\int f(x) dx = \{H : H'(x) = f(x) \forall x \in I\} = F(x) + c$$

Remark. Let $F(x) = \begin{cases} \ln x & \text{if } x > 0 \\ 4 + \ln(-x) & \text{if } x < 0 \end{cases}$ and $G(x) = \begin{cases} 3 + \ln x & \text{if } x > 0 \\ -2 + \ln(-x) & \text{if } x < 0 \end{cases}$

$$\implies F'(x) = G'(x) = \frac{1}{x} \quad \forall x \in \mathbb{R} \setminus \{0\} = H$$

$\implies F$ and G are antiderivatives of $f(x) = \frac{1}{x}$ on the set H , however,

$$F(x) - G(x) = \begin{cases} -3 & \text{if } x > 0 \\ 6 & \text{if } x < 0 \end{cases}, \text{ so their difference is not a constant.}$$

It is important, that the above theorem holds only on an interval.

On the contrary, we use the following notation: $\int \frac{1}{x} dx = \ln |x| + c$

It means that $\int \frac{1}{x} dx = \ln x + c$ if $I \subset (0, \infty)$ and $\int \frac{1}{x} dx = \ln(-x) + c$ if $I \subset (-\infty, 0)$.

Theorem. If f and g have antiderivatives on I and $c \in \mathbb{R}$ then
 $f + g$, $f - g$ and cf also have antiderivatives on I and

1) $\int (f \pm g) = \int f \pm \int g$

2) $\int cf = c \int f$

Basic integrals

$$\int x^\alpha dx = \frac{x^{\alpha+1}}{\alpha+1} + c \quad (\alpha \neq -1) \qquad \int \frac{1}{x} dx = \ln |x| + c$$

$$\int e^x dx = e^x + c \qquad \int a^x dx = \frac{a^x}{\ln a} + c \quad (0 < a \neq 1)$$

$$\int \sin x dx = -\cos x + c \qquad \int \cos x dx = \sin x + c$$

$$\int \frac{1}{\cos^2 x} dx = \tan x + c \qquad \int \frac{1}{\sin^2 x} dx = -\cot x + c$$

$$\int \frac{1}{\sqrt{1-x^2}} dx = \arcsin x + c \qquad \int \frac{1}{1+x^2} dx = \arctan x + c$$

$$\qquad \qquad \qquad = -\arccos x + c \qquad \qquad \qquad = -\operatorname{arccot} x + c$$

$$\int \sinh x dx = \cosh x + c \qquad \int \cosh x dx = \sinh x + c$$

$$\int \frac{1}{\cosh^2 x} dx = \tanh x + c \qquad \int \frac{1}{\sinh^2 x} dx = -\operatorname{coth} x + c$$

$$\int \frac{1}{\sqrt{x^2+1}} dx = \operatorname{arsinh} x + c \qquad \int \frac{1}{\sqrt{x^2-1}} dx = \operatorname{arcosh} x + c$$

$$\int \frac{1}{1-x^2} dx = \frac{1}{2} \ln \left| \frac{1+x}{1-x} \right| + c$$

Integration methods

Theorem. $\int f(ax+b) dx = \frac{F(ax+b)}{a} + c$, where $F' = f$ and $a \neq 0$

Proof. $\left(\frac{F(ax+b)}{a} + c \right)' = \frac{1}{a} F'(ax+b) \cdot a = f(ax+b)$

Example 1. $\int \sqrt{5x-8} dx = \int (5x-8)^{\frac{1}{2}} dx = \frac{(5x-8)^{\frac{3}{2}}}{\frac{3}{2} \cdot 5} + c$

Example 2. $\int (e^{-x} + \cos 2x) dx = \frac{e^{-x}}{-1} + \frac{\sin 2x}{2} + c$

Example 3. $\int \frac{5}{4x^2+1} dx = \int 5 \cdot \frac{1}{(2x)^2+1} dx = 5 \cdot \frac{\operatorname{arctg}(2x)}{2} + c$

Example 4. $\int \frac{5}{4x^2+2} dx = \int \frac{5}{2} \cdot \frac{1}{2x^2+1} dx = \int \frac{5}{2} \cdot \frac{1}{(\sqrt{2}x)^2+1} dx = \frac{5}{2} \cdot \frac{\operatorname{arctg}(\sqrt{2}x)}{\sqrt{2}} + c$

Theorem. 1) $\int \frac{f'(x)}{f(x)} dx = \ln |f(x)| + c$

2) $\int f'(x) (f(x))^\alpha dx = \frac{(f(x))^{\alpha+1}}{\alpha+1} + c, \quad \alpha \neq -1$

Proof. These are consequences of the differentiation rules.

$$\text{Example 1. } \int \frac{x}{x^2+1} dx = \frac{1}{2} \int \frac{2x}{x^2+1} dx = \frac{1}{2} \ln(x^2+1) + c$$

$$\text{Example 2. } \int \tan x dx = - \int \frac{-\sin x}{\cos x} dx = -\ln |\cos x| + c$$

$$\text{Example 3. } \int \cos x \sin^3 x dx = \frac{\sin^4 x}{4} + c \quad (f(x) = \sin x, f'(x) = \cos x, \alpha = 3)$$

$$\text{Example 4. } \int x \sqrt{1+x^2} dx = \frac{1}{2} \cdot \int 2x(1+x^2)^{\frac{1}{2}} dx = \frac{1}{2} \cdot \frac{(1+x^2)^{\frac{3}{2}}}{\frac{3}{2}} + c = \frac{1}{3} \cdot \sqrt{(1+x^2)^3} + c$$

Integration by parts

Theorem. Assume that f and g are differentiable on the interval I and $f \cdot g'$ has an antiderivative on I . Then $f' \cdot g$ also has an antiderivative here and

$$\int f'(x) g(x) dx = f(x) g(x) - \int f(x) g'(x) dx$$

Proof. The right-hand side is differentiable and its derivative is

$$(f(x) g(x) - \int f(x) g'(x) dx)' = f'(x) g(x) + f(x) g'(x) - f(x) g'(x) = f'(x) g(x).$$

Applications

1. $g(x)$ is a polynomial of degree n and

$$f'(x) = e^{ax+b}, \sin(ax+b), \cos(ax+b), \sinh(ax+b), \cosh(ax+b)$$

\Rightarrow the method has to be applied n times

Example 1.

$$\int x \cos 2x dx = \frac{\sin 2x}{2} \cdot x - \int \frac{\sin 2x}{2} \cdot 1 dx = x \cdot \frac{\sin 2x}{2} + \frac{\cos 2x}{4} + c$$

$$f'(x) = \cos 2x \Rightarrow f(x) = \frac{\sin 2x}{2}$$

$$g(x) = x \Rightarrow g'(x) = 1$$

Example 2.

$$\begin{aligned} \int x^2 e^x dx &= e^x \cdot x^2 - \int e^x \cdot 2x dx = e^x \cdot x^2 - (e^x \cdot 2x - \int e^x \cdot 2 dx) = \\ &= e^x \cdot x^2 - e^x \cdot 2x + 2e^x + c \end{aligned}$$

$$f'(x) = e^x \Rightarrow f(x) = e^x \quad u'(x) = e^x \Rightarrow u(x) = e^x$$

$$g(x) = x^2 \Rightarrow g'(x) = 2x \quad v(x) = 2x \Rightarrow v'(x) = 2$$

2. $f'(x)$ is a polynomial and

$$g(x) = \ln x, \arcsin x, \arccos x, \arctan x, \operatorname{arccot} x, \operatorname{arsinh} x, \dots$$

Example 1.

$$\int \ln x \, dx = \int 1 \cdot \ln x \, dx = x \ln x - \int x \cdot \frac{1}{x} \, dx = x \ln x - \int 1 \, dx = x \ln x - x + c$$

$$f'(x) = 1 \implies f(x) = x$$

$$g(x) = \ln x \implies g'(x) = \frac{1}{x}$$

Example 2.

$$\int x \ln x \, dx = \frac{x^2}{2} \ln x - \int \frac{x^2}{2} \cdot \frac{1}{x} \, dx = \frac{x^2}{2} \ln x - \int \frac{x}{2} \, dx = \frac{x^2}{2} \ln x - \frac{x^2}{4} + c$$

$$f'(x) = x \implies f(x) = \frac{x^2}{2}$$

$$g(x) = \ln x \implies g'(x) = \frac{1}{x}$$

Example 3.

$$\int \arctg x \, dx = \int 1 \cdot \arctg x \, dx = x \cdot \arctg x - \int x \cdot \frac{1}{1+x^2} \, dx$$

$$= x \cdot \arctg x - \frac{1}{2} \int \frac{2x}{1+x^2} \, dx = x \cdot \arctg x - \frac{1}{2} \ln(1+x^2) + c$$

$$f'(x) = 1 \implies f(x) = x$$

$$g(x) = \arctg x \implies g'(x) = \frac{1}{1+x^2}$$

$$\int \frac{f'(x)}{f(x)} \, dx = \ln |f(x)| + c$$

Example 4.

$$\int \arcsin x \, dx = \int 1 \cdot \arcsin x \, dx = x \cdot \arcsin x - \int x \cdot \frac{1}{\sqrt{1-x^2}} \, dx$$

$$= x \cdot \arcsin x - \int x \cdot (1-x^2)^{-\frac{1}{2}} \, dx = x \cdot \arcsin x + \frac{1}{2} \cdot \int (-2x) (1-x^2)^{-\frac{1}{2}} \, dx =$$

$$= x \cdot \arcsin x + \frac{1}{2} \frac{(1-x^2)^{\frac{1}{2}}}{-\frac{1}{2}} + c = x \arcsin x + \sqrt{1-x^2} + c$$

$$f'(x) = 1 \implies f(x) = x$$

$$g(x) = \arcsin x \implies g'(x) = \frac{1}{\sqrt{1-x^2}}$$

$$\int f'(x) \cdot (f(x))^\alpha \, dx = \frac{(f(x))^{\alpha+1}}{\alpha+1} + c, \alpha \neq 0$$

3. $f'(x)$ and $g(x)$ are both one of the following functions:

e^{ax+b} , $\sin(ax+b)$, $\cos(ax+b)$, $\sinh(ax+b)$, $\cosh(ax+b)$

\implies the method has to be applied twice

Example. $I = \int e^x \sin x \, dx = ?$

$$I = \int e^x \sin x \, dx = e^x \sin x - \int e^x \cos x \, dx$$

$$f'(x) = e^x \implies f(x) = e^x$$

$$g(x) = \sin x \implies g'(x) = \cos x$$

$$I = \int e^x \sin x \, dx = e^x(-\cos x) - \int e^x(-\cos x) \, dx$$

$$f'(x) = \sin x \implies f(x) = -\cos x$$

$$g(x) = e^x \implies g'(x) = e^x$$

$$(1) \quad I = e^x \sin x - \int e^x \cos x \, dx$$

$$(2) \quad I = e^x(-\cos x) + \int e^x \cos x \, dx$$

$$(1) + (2) \implies 2I = e^x \sin x - e^x \cos x \implies I = \frac{1}{2} e^x (\sin x - \cos x) + c$$

Powers of $\sin x$ and $\cos x$

Odd powers of $\sin x$ and $\cos x$:

$$\sin^{2n+1} x = \sin x \cdot (\sin^2 x)^n = \sin x \cdot (1 - \cos^2 x)^n = \dots$$

$$\cos^{2n+1} x = \cos x \cdot (\cos^2 x)^n = \cos x \cdot (1 - \sin^2 x)^n = \dots$$

Even powers of $\sin x$ and $\cos x$:

$$\sin^{2n} x = (\sin^2 x)^n = \left(\frac{1 - \cos 2x}{2} \right)^n + \dots$$

$$\cos^{2n} x = (\cos^2 x)^n = \left(\frac{1 + \cos 2x}{2} \right)^n + \dots$$

Example 1.

$$I = \int \sin^3 x \, dx = \int \sin x \sin^2 x \, dx = \int \sin x (1 - \cos^2 x) \, dx =$$

$$= \int (\sin x - \sin x \cos^2 x) \, dx = -\cos x + \frac{\cos^3 x}{3} + c$$

$$\int f'(x) \cdot (f(x))^\alpha \, dx = \frac{(f(x))^{\alpha+1}}{\alpha+1} + c, \quad \alpha \neq 0$$

Example 2.

$$I = \int \cos^5 x \, dx = \int \cos^4 x \cos x \, dx = \int (1 - \sin^2 x)^2 \cos x \, dx =$$

$$= \int (1 - 2\sin^2 x + \sin^4 x) \cos x \, dx = \int (\cos x - 2\sin^2 x \cos x + \sin^4 x \cos x) \, dx$$

$$= \sin x - \frac{2}{3} \sin^3 x + \frac{\sin^5 x}{5} + c$$

$$\int f'(x) \cdot (f(x))^\alpha \, dx = \frac{(f(x))^{\alpha+1}}{\alpha+1} + c, \quad \alpha \neq 0$$

Example 3.

$$I = \int \sin^2 x \, dx = \int \frac{1 - \cos 2x}{2} \, dx = \frac{x}{2} - \frac{\sin 2x}{4} + c$$

Example 4.

$$\begin{aligned}
 I &= \int \cos^4 x \, dx = \int (\cos^2 x)^2 \, dx = \int \left(\frac{1 + \cos 2x}{2} \right)^2 \, dx = \\
 &= \int \frac{1}{4} \cdot (1 + 2 \cos 2x + \cos^2 2x) \, dx = \\
 &\qquad \qquad \qquad \cos^2 2x = \frac{1 + \cos 4x}{2} \\
 &= \int \left(\frac{1}{4} + \frac{1}{2} \cos 2x + \frac{1}{8} + \frac{1}{8} \cos 4x \right) \, dx = \frac{3}{8} x + \frac{1}{4} \sin 2x + \frac{1}{32} \sin 4x + c
 \end{aligned}$$

Products of powers of $\sin x$ and $\cos x$:

$$\int \sin^n x \cos^m x \, dx = ? \quad n, m \in \mathbb{N}^+$$

1. If n or m is odd:

$$\begin{aligned}
 \int \sin^3 x \cos^4 x \, dx &= \int \sin x \sin^2 x \cos^4 x \, dx = \int \sin x (1 - \cos^2 x) \cos^4 x \, dx = \\
 &= \int \sin x (\cos^4 x - \cos^6 x) \, dx = -\frac{\cos^5 x}{5} + \frac{\cos^7 x}{7} + c
 \end{aligned}$$

2. If n and m are even:

$$\int \sin^2 x \cos^4 x \, dx = \int (1 - \cos^2 x) \cos^4 x \, dx = \int (\cos^4 x - \cos^6 x) \, dx = \dots$$

Integrals of rational functions

Polynomial division

Definition: If $p_1(x)$ and $p_2(x)$ are polynomials then the function $R(x) = \frac{p_1(x)}{p_2(x)}$

is called a rational function.

Statement. Polynomials can be divided in the following sense:

$$p_1(x) = q(x)p_2(x) + r(x)$$

where $\deg r < \deg p_2$. If $\deg p_1 < \deg p_2$ then $q = 0$ and $r = p_1$.

The polynomial q is the quotient and r is the remainder.

Example 1. Divide $p_1(x) = x^3 - 4x^2 + 5x - 8$ by $p_2(x) = x^2 + 3$

$$\begin{array}{r}
 (x^3 - 4x^2 + 5x - 8) : (x^2 + 3) = x - 4 \quad (x^3 : x^2 = x, \text{ then multiply } p_2(x) \text{ by } x^2) \\
 -(x^3 \qquad + 3x) \\
 \hline
 \end{array}$$

$$\begin{array}{r}
 0 - 4x^2 + 2x - 8 \qquad \qquad (-4x^2 : x^2 = -4, \text{ then multiply } p_2(x) \text{ by } -4) \\
 -(-4x^2 \qquad - 12) \\
 \hline
 \end{array}$$

$$0 + 2x + 4 \qquad \Rightarrow \text{the remainder is } 2x + 4, \text{ since } \deg(2x + 4) < \deg(x^2 + 3)$$

$$\Rightarrow x^3 - 4x^2 + 5x - 8 = (x^2 + 3)(x - 4) + (2x + 4)$$

$$\Rightarrow \frac{x^3 - 4x^2 + 5x - 8}{x^2 + 3} = (x - 4) + \frac{2x + 4}{x^2 + 3}$$

Example 2. Divide $p_1(x) = x^4$ by $p_2(x) = x^2 + x - 2$

$$\begin{array}{r} x^4 \\ -(x^4 + x^3 - 2x^2) \\ \hline \end{array} : (x^2 + x - 2) = x^2 - x + 3 \quad (x^4 : x^2 = x^2, \text{ then multiply } p_2(x) \text{ by } x^2)$$

(the product is subtracted from the line above)

$$\begin{array}{r} 0 - x^3 + 2x^2 \\ -(-x^3 - x^2 + 2x) \\ \hline \end{array} \quad (-x^3 : x^2 = -x, \text{ then multiply } p_2(x) \text{ by } -x)$$

$$\begin{array}{r} 0 + 3x^2 - 2x \\ -(3x^2 + 3x - 6) \\ \hline \end{array} \quad (3x^2 : x^2 = 3, \text{ then multiply } p_2(x) \text{ by } 3)$$

$$0 - 5x + 6 \quad \Rightarrow \text{the remainder is } -5x + 6, \text{ since } \deg(-5x + 6) < \deg(x^2 - x + 3)$$

$$\Rightarrow x^4 = (x^2 + x - 2)(x^2 - x + 3) + (-5x + 6)$$

$$\Rightarrow \frac{x^4}{x^2 + x - 2} = (x^2 - x + 3) + \frac{-5x + 6}{x^2 + x - 2}$$

Integration of rational functions

1st step. If $R(x) = \frac{T(x)}{Q(x)}$ and $\deg T(x) \geq \deg Q(x)$ then with polynomial division

we bring it to the form $R(x) = E(x) + \frac{P(x)}{Q(x)}$, where $E(x)$ is a polynomial

and $\deg P(x) < \deg Q(x)$.

2nd step. The denominator can be written as

$$Q(x) = (x - a_1)^{\alpha_1} \dots (x - a_r)^{\alpha_r} (x^2 + b_1x + c_1)^{\beta_1} \dots (x^2 + b_sx + c_s)^{\beta_s}$$

where $b_i^2 - 4c_i < 0$ and $\alpha_1, \dots, \alpha_r$ are the multiplicities of the real roots

and β_1, \dots, β_s are the multiplicities of the complex roots.

3rd step. Partial fraction decomposition. It means that to each term in the above form of $Q(x)$ we assign an elementary fraction (or partial fraction) such that the sum of these fractions is equal to $\frac{P(x)}{Q(x)}$. This decomposition is unique.

	Factor in the denominator	Term in the partial fraction decomposition
Single real root :	$x - a$	$\frac{A}{x - a}$
Multiple real root :	$(x - a)^k$	$\frac{A_1}{x - a} + \frac{A_2}{(x - a)^2} + \dots + \frac{A_k}{(x - a)^k}$
Single complex roots ($b^2 - 4c < 0$):	$x^2 + bx + c$	$\frac{Bx + C}{x^2 + bx + c}$
Multiple complex roots ($b^2 - 4c < 0$):	$(x^2 + bx + c)^k$	$\frac{B_1x + C_1}{x^2 + bx + c} + \frac{B_2x + C_2}{(x^2 + bx + c)^2} + \dots + \frac{B_kx + C_k}{(x^2 + bx + c)^k}$

4th step. We integrate the polynomial $E(x)$ and the partial fractions term by term.

Integration of the elementary fractions

$$(1) \int \frac{A}{x-a} dx = A \ln |x-a| + c$$

$$(2) k \in \mathbb{N}, k > 1: \int \frac{A}{(x-1)^k} dx = \frac{-kA}{(x-1)^{k-1}} + c$$

$$(3) b^2 - 4c < 0: \int \frac{Bx+C}{x^2+bx+c} dx = \int \frac{B}{2} \frac{2x+b}{x^2+bx+c} dx + \int \frac{-\frac{Bb}{2}+C}{x^2+bx+c} dx$$

$$(i) \int \frac{2x+b}{x^2+bx+c} dx = \ln(x^2+bx+c) + \text{constant}$$

$$(ii) \int \frac{1}{x^2+bx+c} dx = \int \frac{1}{\left(x+\frac{b}{2}\right)^2+d^2} dx = \frac{1}{d^2} \int \frac{1}{\left(\frac{x+\frac{b}{2}}{d}\right)^2+1} dx = \frac{1}{d^2} \frac{\arctan\left(\frac{x+\frac{b}{2}}{d}\right)}{\frac{1}{d}} + \text{constant}$$

$$\text{where } d = \sqrt{c - \frac{b^2}{4}}.$$

$$(4) b^2 - 4c < 0, k \in \mathbb{N}, k > 1: \int \frac{Bx+C}{(x^2+bx+c)^k} dx = \int \frac{B}{2} \frac{2x+b}{(x^2+bx+c)^k} dx + \int \frac{-\frac{Bb}{2}+C}{(x^2+bx+c)^k} dx$$

$$(i) \int \frac{2x+b}{(x^2+bx+c)^k} dx = \frac{1}{1-k} \frac{1}{(x^2+bx+c)^{k-1}} + \text{constant}$$

$$(ii)^* \int \frac{1}{(x^2+bx+c)^k} dx = \int \frac{1}{\left(\left(x+\frac{b}{2}\right)^2+d^2\right)^k} dx, \text{ where } d = \sqrt{c - \frac{b^2}{4}}$$

$$\text{If } \int \frac{1}{(x^2+1)^n} dx = F_n(x) + c \text{ then } \int \frac{1}{\left(\left(x+\frac{b}{2}\right)^2+d^2\right)^k} dx = \frac{1}{d^{2n-1}} F_n\left(\frac{x}{d} + \frac{b}{2d}\right) + c.$$

Remark. If $\int \frac{1}{(x^2+1)^n} dx = F_n(x) + c$ then for $n \geq 1$: $F_{n+1}(x) = \frac{1}{2n} \frac{x}{(x^2+1)^n} + \frac{2n-1}{2n} F_n(x) + c$.

For example, $F_1(x) = \arctan x + c$

$$F_2(x) = \frac{1}{2} \frac{x}{x^2+1} + \frac{1}{2} \arctan x + c$$

$$F_3(x) = \frac{1}{4} \frac{x}{x^2+1} + \frac{3}{8} \frac{x}{x^2+1} + \frac{3}{8} \arctan x + c$$

Examples

Example 1. $I = \int \frac{x+1}{x^2+3x} dx = ?$

Solution. The denominator has two distinct real roots: $x_1 = 0$, $x_2 = -3$.

Partial fraction decomposition:

$$\frac{x+1}{x^2+3x} = \frac{x+1}{x(x+3)} = \frac{A}{x} + \frac{B}{x+3} = \frac{A(x+3) + Bx}{x(x+3)}$$

$$\Rightarrow x + 1 = A(x + 3) + Bx$$

We want to find A and B such that this equation holds **for all** $x \in \mathbb{R}$.

1st method (comparison of the coefficients)

$x + 1 = (A + B)x + 3A$ holds **for all** $x \in \mathbb{R}$ if and only if

$$\begin{aligned} A + B = 1 &\Rightarrow A = \frac{1}{3}, B = \frac{2}{3} \\ 3A = 1 & \end{aligned}$$

2nd method (substitution) (it is worth substituting the roots of the denominator)

$$x + 1 = A(x + 3) + Bx$$

$$\text{if } x = 0 \Rightarrow 1 = A \cdot 3 + B \cdot 0 \Rightarrow A = \frac{1}{3}$$

$$\text{if } x = -3 \Rightarrow -2 = A \cdot 0 + B \cdot (-3) \Rightarrow B = \frac{2}{3}$$

$$\Rightarrow I = \int \frac{x+1}{x^2+3x} dx = \int \left(\frac{1}{3} \cdot \frac{1}{x} + \frac{2}{3} \cdot \frac{1}{x+3} \right) dx = \frac{1}{3} \ln |x| + \frac{2}{3} \ln |x+3| + c$$

Example 2. $I = \int \frac{x+5}{x^2+6x+9} dx = ?$

Solution. The denominator has multiple real roots: $x_{1,2} = -3$.

Partial fraction decomposition:

$$\frac{x+5}{x^2+6x+9} = \frac{x+5}{(x+3)^2} = \frac{A}{x+3} + \frac{B}{(x+3)^2} = \frac{A(x+3)+B}{(x+3)^2}$$

$$\Rightarrow x + 5 = A(x + 3) + B$$

1st method: $x + 5 = Ax + (3A + B) \Rightarrow A = 1 \quad \Rightarrow A = 1, B = 2$

$$3A + B = 5$$

2nd method: $x + 5 = A(x + 3) + B$

$$x = -3 \Rightarrow 2 = 0 + B \Rightarrow B = 2$$

$$x = 0 \Rightarrow 5 = 3A + B \Rightarrow A = 1$$

$$\begin{aligned} \Rightarrow I &= \int \frac{x+5}{x^2+6x+9} dx = \int \left(\frac{1}{x+3} + \frac{2}{(x+3)^2} \right) dx = \int \left(\frac{1}{x+3} + 2(x+3)^{-2} \right) dx = \\ &= \ln |x+3| + 2 \frac{(x+3)^{-1}}{-1} + c = \ln |x+3| - \frac{2}{x+3} + c \end{aligned}$$

Example 3. $I = \int \frac{1}{(x-1)^2(x^2+1)} dx = ?$

Solution: The roots of the denominator are:

$$x_{1,2} = 1 \text{ (multiple real roots), } x_{3,4} = \pm i \text{ (simple complex roots)}$$

Partial fraction decomposition:

$$\frac{1}{(x-1)^2(x^2+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+1} = \frac{A(x-1)(x^2+1) + B(x^2+1) + (Cx+D)(x-1)^2}{(x-1)^2(x^2+1)}$$

$$\Rightarrow 1 = A(x-1)(x^2+1) + B(x^2+1) + (Cx+D)(x-1)^2$$

Substitutions:

$$x = 1 \Rightarrow 1 = 0 + B + 0 \Rightarrow B = 1$$

$$x = 0 \Rightarrow 1 = -A + B + D$$

$$x = -1 \Rightarrow 1 = -4A + 2B - 4C + 4D$$

$$x = 2 \Rightarrow 1 = 5A + 5B + 2C + D$$

Homework: the solution of this equation system is

$$C = \frac{1}{2}, D = 0, A = -\frac{1}{2}, B = \frac{1}{2}$$

$$\begin{aligned} \Rightarrow I &= \int \frac{1}{(x-1)^2(x^2+1)} dx = \int \left(-\frac{1}{2} \cdot \frac{1}{x-1} + \frac{1}{2} \cdot \frac{1}{(x-1)^2} + \frac{1}{2} \cdot \frac{x}{x^2+1} \right) dx = \\ &= \int \left(-\frac{1}{2} \cdot \frac{1}{x-1} + \frac{1}{2} \cdot (x-1)^{-2} + \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{2x}{x^2+1} \right) dx = \left(\text{the last term has the form } \frac{f'}{f} \right) \\ &= -\frac{1}{2} \ln |x-1| + \frac{1}{2} \cdot \frac{(x-1)^{-1}}{-1} + \frac{1}{4} \ln(x^2+1) + c \end{aligned}$$

Integration by change of variables

The substitution formula

Theorem. Assume that g is differentiable on the interval I , f is defined on $J = g(I)$ and f has a primitive function on J . Then $(f \circ g)g'$ also has a primitive function on I and

$$\int f(g(x)) \cdot g'(x) dx = F(g(x)) + c \quad \text{where} \quad \int f(x) dx = F(x) + c$$

Proof. $(F(g(x)) + c)' = F'(g(x)) \cdot g'(x) = f(g(x)) \cdot g'(x)$

Remark. If g is invertible then the above formula can be written in the form

$$\int f(g(t)) \cdot g'(t) dt \Big|_{t=g^{-1}(x)} = \int f(x) dx$$

Examples

Example 1. $I = \int \sin x e^{\cos x} dx = ?$

Substitution: $t = \cos x \Rightarrow x = x(t) = \arccos t$

$$x'(t) = \frac{dx}{dt} = -\frac{1}{\sqrt{1-t^2}} \Rightarrow dx = -\frac{1}{\sqrt{1-t^2}} dt$$

$$\sin x = \sqrt{\sin^2 x} = \sqrt{1 - \cos^2 x} = \sqrt{1 - t^2}$$

$$I = \int \sin x e^{\cos x} dx = \int \sqrt{1-t^2} e^t \cdot \left(-\frac{1}{\sqrt{1-t^2}}\right) dt = \int -e^t dt = -e^t + c = -e^{\cos x} + c$$

Remark. $\int e^{f(x)} \cdot f'(x) dx = e^{f(x)} + c$

Example 2. $I = \int x^2 \sin x^3 dx = ?$

Substitution: $t = x^3 \Rightarrow x = x(t) = \sqrt[3]{t} = t^{\frac{1}{3}} \Rightarrow x^2 = t^{\frac{2}{3}}$

$$x'(t) = \frac{dx}{dt} = \frac{1}{3} t^{-\frac{2}{3}} \Rightarrow dx = \frac{1}{3} t^{-\frac{2}{3}} dt$$

$$I = \int x^2 \sin(x^3) dx = \int t^{\frac{2}{3}} \sin t \cdot \frac{1}{3} t^{-\frac{2}{3}} dt = \int \frac{1}{3} \sin t dt = -\frac{1}{3} \cos t + c = -\frac{1}{3} \cos x^3 + c$$

Example 3. $I = \int \sin^4 x \cos x dx = ?$

1st solution. $\int f'(x) f^\alpha(x) dx = \frac{f^{\alpha+1}(x)}{\alpha+1} + c \quad (\alpha \neq -1)$

Here: $f(x) = \sin x$, $f'(x) = \cos x$, $\alpha = 4 \Rightarrow$

$$I = \int \sin^4 x \cos x dx = \frac{\sin^5 x}{5} + c$$

2nd solution.

Substitution: $t = \sin x \Rightarrow x = x(t) = \arcsin t$

$$x'(t) = \frac{dx}{dt} = \frac{1}{\sqrt{1-t^2}} \Rightarrow dx = \frac{1}{\sqrt{1-t^2}} dt$$

$$\cos x = \sqrt{\cos^2 x} = \sqrt{1 - \sin^2 x} = \sqrt{1 - t^2}$$

$$I = \int \sin^4 x \cos x dx = \int t^4 \sqrt{1-t^2} \cdot \frac{1}{\sqrt{1-t^2}} dt = \int t^4 dt = \frac{t^5}{5} + c = \frac{\sin^5 x}{5} + c$$

Rational functions of e^x

Statement. The integral $\int R(e^x) dx$ (where R is a rational function) can be transformed to the integral of a rational function $t = e^x$.

Example 1. $I = \int \frac{e^{2x}}{e^x + 1} dx = ?$

Substitution: $t = e^x \Rightarrow x = x(t) = \ln t$

$$x'(t) = \frac{dx}{dt} = \frac{1}{t} \Rightarrow dx = \frac{1}{t} dt$$

$$\begin{aligned} I &= \int \frac{e^{2x}}{e^x + 1} dx = \int \frac{t^2}{t+1} \cdot \frac{1}{t} dt = \int \frac{t}{t+1} dt = \int \frac{(t+1)-1}{t+1} dt = \int \left(1 - \frac{1}{t+1}\right) dt = \\ &= t - \ln |t+1| + c = e^x - \ln(e^x + 1) + c \end{aligned}$$

Example 2. $I = \int \frac{4}{e^{2x} - 4} dx = ?$

Substitution: $t = e^x \Rightarrow x = x(t) = \ln t$

$$x'(t) = \frac{dx}{dt} = \frac{1}{t} \Rightarrow dx = \frac{1}{t} dt$$

$$I = \int \frac{4}{e^{2x} - 4} dx = \int \frac{4}{t^2 - 4} \cdot \frac{1}{t} dt = \int \frac{4}{t(t-2)(t+2)} dt$$

Partial fraction decomposition:

$$\frac{4}{t(t-2)(t+2)} = \frac{A}{t} + \frac{B}{t+2} + \frac{C}{t-2}$$

$$\Rightarrow 4 = A(t+2)(t-2) + Bt(t-2) + Ct(t+2)$$

$$t = 0: 4 = -4A + 0 + 0 \Rightarrow A = -1$$

$$t = -2: 4 = 0 + 8B + 0 \Rightarrow B = \frac{1}{2}$$

$$t = 2: 4 = 0 + 0 + 8C \Rightarrow C = \frac{1}{2}$$

$$\begin{aligned} \Rightarrow I &= \int \left(-\frac{1}{t} + \frac{1}{2} \frac{1}{t+2} + \frac{1}{2} \frac{1}{t-2} \right) dt = -\ln |t| + \frac{1}{2} \ln |t+2| + \frac{1}{2} \ln |t-2| + c = \\ &= -\ln e^x + \frac{1}{2} \ln(e^x + 2) + \frac{1}{2} \ln |e^x - 2| + c \end{aligned}$$

Some integrals with roots

Remark. In the following cases $R(u, v)$ denotes a two-variable rational function,

$$\text{that is, } R(u, v) = \frac{P(u, v)}{Q(u, v)}, \quad P(u, v) = \sum_{i=1}^n \sum_{j=1}^n a_{ij} u^i v^j, \quad Q(u, v) = \sum_{i=1}^n \sum_{j=1}^n b_{ij} u^i v^j, \quad n \in \mathbf{N}.$$

1. The integral $\int R\left(x, \sqrt[n]{\frac{ax+b}{cx+d}}\right) dx$ (where $c^2 + d^2 \neq 0$, $ad \neq bc$) can be transformed

to the integral of a rational function with the substitution $t = \sqrt[n]{\frac{ax+b}{cx+d}}$.

Example 1. $I = \int \frac{1}{x^2} \sqrt[3]{\frac{x+1}{x}} dx = ?$

$$\text{Substitution: } t = \sqrt[3]{\frac{x+1}{x}} \Rightarrow x+1 = t^3 x$$

$$x = x(t) = \frac{1}{t^3 - 1}$$

$$x'(t) = \frac{dx}{dt} = -\frac{3t^2}{(t^3 - 1)^2} \Rightarrow dx = -\frac{3t^2}{(t^3 - 1)^2} dt$$

$$I = \int \frac{1}{x^2} \sqrt[3]{\frac{x+1}{x}} dx = \int (t^3 - 1)^2 \cdot t \cdot \frac{-3t^2}{(t^3 - 1)^2} dt = \int -3t^3 dt = -\frac{3}{4} t^4 + c = -\frac{3}{4} \left(\frac{x+1}{x}\right)^{\frac{4}{3}} + c$$

Example 2. $I = \int \sqrt{\frac{x-3}{x-1}} dx = ?$

$$\text{Substitution: } t = \sqrt{\frac{x-3}{x-1}} \Rightarrow t^2(x-1) = x-3 \Rightarrow x(t^2 - 1) = t^2 - 3$$

$$x = x(t) = \frac{t^2 - 3}{t^2 - 1} = \frac{t^2 - 1 - 2}{t^2 - 1} = 1 - \frac{2}{t^2 - 1}$$

$$x'(t) = \frac{dx}{dt} = \frac{4t}{(t^2 - 1)^2} \Rightarrow dx = \frac{4t}{(t^2 - 1)^2} dt$$

$$I = \int \sqrt{\frac{x-3}{x-1}} dx = \int t \cdot \frac{4t}{(t^2 - 1)^2} dt = \int \frac{4t^2}{(t-1)^2 (t+1)^2} dt$$

$$\text{Partial fraction decomposition: } \frac{4t^2}{(t-1)^2 (t+1)^2} = \frac{A}{t-1} + \frac{B}{(t-1)^2} + \frac{C}{t+1} + \frac{D}{(t+1)^2}$$

$$\Rightarrow A = 1, B = 1, C = -1, D = 1$$

$$\Rightarrow I = \int \left(\frac{1}{t-1} + \frac{1}{(t-1)^2} - \frac{1}{t+1} + \frac{1}{(t+1)^2} \right) dt = \ln |t-1| - \frac{1}{t-1} - \ln |t+1| - \frac{1}{t+1} + c$$

$$= -\frac{2t}{t^2 - 1} + \ln \left| \frac{t-1}{t+1} \right| + c = -\frac{2\sqrt{\frac{x-3}{x-1}}}{\frac{x-3}{x-1} - 1} + \ln \left| \frac{\sqrt{\frac{x-3}{x-1}} - 1}{\sqrt{\frac{x-3}{x-1}} + 1} \right| + c$$

2. The integral $\int R\left(x, \sqrt[n]{ax+b}\right) dx$ can be transformed to the integral of a rational function with the substitution $t = \sqrt[n]{ax+b}$.

Example 1. $I = \int x \sqrt{5x+3} \, dx = ?$

$$\text{Substitution: } t = \sqrt{5x+3} \implies x = x(t) = \frac{t^2 - 3}{5}$$

$$x'(t) = \frac{dx}{dt} = \frac{2}{5}t \implies dx = \frac{2}{5}t \, dt$$

$$\begin{aligned} I &= \int x \sqrt{5x+3} \, dx = \int \frac{t^2 - 3}{5} \cdot t \cdot \frac{2}{5}t \, dt = \int \frac{2}{25} (t^4 - 3t^2) \, dt = \frac{2}{25} \left(\frac{t^5}{5} - t^3 \right) + c = \\ &= \frac{2}{25} \left(\frac{\sqrt{(5x+3)^5}}{5} - \sqrt{(5x+3)^3} \right) + c \end{aligned}$$

Example 2. $I = \int \frac{1}{\sqrt{x+1}} \, dx = ?$

$$\text{Substitution: } t = \sqrt{x} \implies x = x(t) = t^2$$

$$x'(t) = \frac{dx}{dt} = 2t \implies dx = 2t \, dt$$

$$\begin{aligned} I &= \int \frac{1}{\sqrt{x+1}} \, dx = \int \frac{2t}{t+1} \, dt = \int \frac{2(t+1) - 1}{t+1} \, dt = \int 2 \cdot \left(1 - \frac{1}{t+1} \right) dt = \\ &= 2t - 2 \ln |1+t| + c = 2\sqrt{x} - 2 \ln(1 + \sqrt{x}) + c \end{aligned}$$

3. In the integral $\int R(x, \sqrt{ax^2 + bx + c}) \, dx$ ($a \neq 0$) after completing the square under the root sign, either of the following substitutions can be used:

$$\sqrt{1-A^2} \quad A = \sin t, \quad t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \quad (\text{or } A = \cos t, \quad t \in [0, \pi])$$

$$\sqrt{B^2+1} \quad B = \sinh t$$

$$\sqrt{C^2-1} \quad C = \cosh t$$

Identities for taking the square root: $\cos^2 t + \sin^2 t = 1$, $\cosh^2 t - \sinh^2 t = 1$.

Example 1. $I = \int \sqrt{4-x^2} \, dx = ?$

$$\text{Substitution: } x = x(t) = 2 \sin t \implies t = \arcsin\left(\frac{x}{2}\right)$$

$$x'(t) = \frac{dx}{dt} = 2 \cos t \implies dx = 2 \cos t \, dt$$

$$\begin{aligned} I &= \int \sqrt{4-x^2} \, dx = \int \sqrt{4-4\sin^2 t} \cdot 2 \cos t \, dt = \int 2 \sqrt{1-\sin^2 t} \cdot 2 \cos t \, dt = \int 4 \sqrt{\cos^2 t} \cdot \cos t \, dt \\ &= \int 4 \cos^2 t \, dt = \int 4 \cdot \frac{1 + \cos 2t}{2} \, dt = \int 2 \cdot (1 + \cos 2t) \, dt = 2 \left(t + \frac{\sin 2t}{2} \right) + c = \\ &= 2t + \sin 2t + c = 2 \cdot \arcsin\left(\frac{x}{2}\right) + 2 \cdot \frac{x}{2} \sqrt{1 - \left(\frac{x}{2}\right)^2} + c \end{aligned}$$

Identities:

$$\cos^2 x + \sin^2 x = 1 \quad \Rightarrow \quad \cos^2 x = \frac{1 + \cos 2x}{2}, \quad \sin^2 x = \frac{1 - \cos 2x}{2}$$

$$\cos^2 x - \sin^2 x = \cos 2x$$

$$\sin 2t = 2 \sin t \cos t = 2 \sin t \sqrt{\cos^2 t} = 2 \sin t \sqrt{1 - \sin^2 t} = 2 \cdot \frac{x}{2} \cdot \sqrt{1 - \left(\frac{x}{2}\right)^2}$$

Example 2. $I = \int \frac{x^2}{\sqrt{9-x^2}} dx = ?$

Substitution: $x = 3 \sin t \Rightarrow t = \arcsin\left(\frac{x}{3}\right)$

$$x'(t) = \frac{dx}{dt} = 3 \cos t \Rightarrow dx = 3 \cos t dt$$

$$\begin{aligned} I &= \int \frac{x^2}{\sqrt{9-x^2}} dx = \int \frac{(3 \sin t)^2}{\sqrt{9-(3 \sin t)^2}} \cdot 3 \cos t dt = \int \frac{9 \sin^2 t}{\sqrt{9 \cdot (1 - \sin^2 t)}} \cdot 3 \cos t dt = \\ &= \int \frac{9 \sin^2 t}{\sqrt{\cos^2 t}} \cos t dt = \int \frac{9 \sin^2 t}{\cos t} \cos t dt = \int 9 \sin^2 t dt = \int 9 \cdot \frac{1 - \cos 2t}{2} dt = \\ &= \frac{9}{2} \left(t - \frac{\sin 2t}{2} \right) + c = \frac{9}{2} t - \frac{9}{4} \sin 2t + c = \frac{9}{2} \arcsin\left(\frac{x}{3}\right) - \frac{9}{4} \cdot 2 \cdot \frac{x}{3} \sqrt{1 - \left(\frac{x}{3}\right)^2} + c = \end{aligned}$$

Identities:

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

$$\sin 2t = 2 \sin t \cos t = 2 \sin t \sqrt{\cos^2 t} = 2 \sin t \sqrt{1 - \sin^2 t} = 2 \cdot \frac{x}{3} \cdot \sqrt{1 - \left(\frac{x}{3}\right)^2}$$

Rational functions of $\sin x$ and $\cos x$

Statement. The integral $\int R(\sin x, \cos x) dx$ where R is a rational function can be transformed to the integral of a rational function with the substitution

$$t = \tan \frac{x}{2}$$

$$\Rightarrow x = 2 \arctan t, \quad dx = \frac{2}{1+t^2} dt, \quad \sin x = \frac{2t}{1+t^2}, \quad \cos x = \frac{1-t^2}{1+t^2}$$

Remark. $t = \tan \frac{x}{2} \Rightarrow x = x(t) = 2 \arctan t \Rightarrow \frac{dx}{dt} = x'(t) = \frac{2}{1+t^2}$

$$\tan^2 x = \frac{\sin^2 x}{\cos^2 x} = \frac{1 - \cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} - 1 \Rightarrow \cos^2 x = \frac{1}{1 + \tan^2 x}$$

$$\sin x = 2 \sin \frac{x}{2} \cos \frac{x}{2} = 2 \tan \frac{x}{2} \cos^2 \frac{x}{2} = \frac{2 \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} = \frac{2t}{1+t^2}$$

$$\cos x = \cos^2 \frac{x}{2} - \sin^2 \frac{x}{2} = \cos^2 \frac{x}{2} \left(1 - \tan^2 \frac{x}{2} \right) = \frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} = \frac{1-t^2}{1+t^2}$$

Example 1. $I = \int \frac{1}{\sin x} dx = ?$

Substitution: $t = \tan \frac{x}{2} \Rightarrow \sin x = \frac{2t}{1+t^2}, dx = \frac{2}{1+t^2} dt$

$$I = \int \frac{1}{\sin x} dx = \int \frac{1+t^2}{2t} \frac{2}{1+t^2} dt = \int \frac{1}{t} dt = \ln |t| + c = \ln \left| \tan \frac{x}{2} \right| + c$$

Example 2. $I = \int \frac{1}{\cos x} dx = ?$

Substitution: $t = \tan\left(\frac{x}{2} - \frac{\pi}{4}\right)$

$$I = \int \frac{1}{\cos x} dx = \int \frac{1}{\sin\left(\frac{\pi}{2} - x\right)} dx = - \int \frac{1}{\sin\left(x - \frac{\pi}{2}\right)} dx = - \int \frac{1}{t} dt = -\ln \left| \tan\left(\frac{x}{2} - \frac{\pi}{4}\right) \right| + c$$

Example 3. $I = \int \frac{1}{1 + \cos x} dx = ?$

Substitution: $t = \tan \frac{x}{2} \Rightarrow \cos x = \frac{1-t^2}{1+t^2}, dx = \frac{2}{1+t^2} dt$

$$I = \int \frac{1}{1 + \cos x} dx = \int \frac{1}{1 + \frac{1-t^2}{1+t^2}} \frac{2}{1+t^2} dt = \int 1 dt = t + c = \tan \frac{x}{2} + c$$

Example 4. $I = \int \frac{1}{\sin x(1 + \cos x)} dx = ?$

Substitution: $t = \tan \frac{x}{2} \Rightarrow \sin x = \frac{2t}{1+t^2}, \cos x = \frac{1-t^2}{1+t^2}, dx = \frac{2}{1+t^2} dt$

$$\begin{aligned} I &= \int \frac{1}{\sin x(1 + \cos x)} dx = \int \frac{1}{\frac{2t}{1+t^2} \left(1 + \frac{1-t^2}{1+t^2}\right)} \frac{2}{1+t^2} dt = \int \frac{1+t^2}{2t} dt \\ &= \int \left(\frac{1}{2t} + \frac{t}{2}\right) dt = \frac{1}{2} \ln |t| + \frac{t^2}{4} + c = \frac{1}{2} \ln \left| \tan \frac{x}{2} \right| + \frac{1}{4} \tan^2 \frac{x}{2} + c \end{aligned}$$

Example 5*. $I = \int \frac{1 + \sin x}{1 - \cos x} dx = ?$

Substitution: $t = \tan \frac{x}{2} \Rightarrow \sin x = \frac{2t}{1+t^2}, \cos x = \frac{1-t^2}{1+t^2}, dx = \frac{2}{1+t^2} dt$

$$I = \int \frac{1 + \sin x}{1 - \cos x} dx = \int \frac{1 + \frac{2t}{1+t^2}}{1 - \frac{1-t^2}{1+t^2}} \frac{2}{1+t^2} dt = \int \frac{t^2 + 2t + 1}{t^2(t^2 + 1)} dt =$$

Partial fraction decomposition:

$$\frac{t^2 + 2t + 1}{t^2(t^2 + 1)} = \frac{A}{t} + \frac{B}{t^2} + \frac{Ct + D}{t^2 + 1} \Rightarrow A = 2, B = 1, C = -2, D = 0$$

$$I = \int \left(\frac{2}{t} + \frac{1}{t^2} - \frac{2t}{t^2+1} \right) dt = 2 \ln \left| \tan \frac{x}{2} \right| - \frac{1}{\tan \frac{x}{2}} - \ln \left(1 + \tan^2 \frac{x}{2} \right) + c$$

Remark. For even powers of $\sin x$ and $\cos x$ the following transformations are better:

$$1 + \cot^2 x = \frac{\sin^2 x + \cos^2 x}{\sin^2 x} = \frac{1}{\sin^2 x} \implies \sin^2 x = \frac{1}{1 + \cot^2 x}$$

$$1 + \tan^2 x = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} \implies \cos^2 x = \frac{1}{1 + \tan^2 x}$$

Example 1*. $I = \int \frac{1}{\sin^6 x} dx = ?$

$$I = \int \frac{1}{\sin^6 x} dx = \int \left(\frac{1}{\sin^2 x} \right)^2 \frac{1}{\sin^2 x} dx = \int (1 + \cot^2 x)^2 \frac{1}{\sin^2 x} dx$$

Substitution: $y = \cot x \implies \frac{dy}{dx} = -\frac{1}{\sin^2 x}$

$$I = -\int (1 + y^2)^2 dy = -\int (1 + 2y^2 + y^4) dy = -\left(y + \frac{2}{3}y^3 + \frac{y^5}{5} \right) + c$$

$$= -\left(\cot x + \frac{2}{3}(\cot x)^3 + \frac{(\cot x)^5}{5} \right) + c$$

Example 2*. $I = \int \frac{1}{2 + \sin^2 x} dx = ?$

$$I = \int \frac{1}{2 + \sin^2 x} dx = \int \frac{1}{2 + \frac{1}{1 + \cot^2 x}} dx = \int \frac{1 + \cot^2 x}{3 + 2 \cot^2 x} dx$$

Substitution: $y = \cot x$

$$x = \operatorname{arccot} y \implies dx = -\frac{1}{1 + y^2} dy$$

$$I = -\int \frac{1 + y^2}{3 + 2y^2} \frac{1}{1 + y^2} dy = -\int \frac{1}{3 + 2y^2} dy = -\frac{1}{3} \int \frac{1}{1 + \left(\sqrt{\frac{2}{3}} y \right)^2} dy$$

$$= -\frac{1}{3} \frac{\arctan\left(\sqrt{\frac{2}{3}} y \right)}{\sqrt{\frac{2}{3}}} + c = -\frac{1}{\sqrt{6}} \arctan\left(\sqrt{\frac{2}{3}} \cot x \right) + c$$

Additional examples

Substitution

Example 1. $I = \int \tan^6 x dx = ?$

Substitution: $y = \tan x \implies x = x(t) = \arctan y$

$$x'(y) = \frac{dx}{dy} = \frac{1}{1 + y^2} \implies dx = \frac{1}{1 + y^2} dy$$

$$I = \int \tan^6 x \, dx = \int \frac{y^6}{y^2 + 1} \, dy$$

Here y^6 has to be divided by $y^2 + 1$. This can be done in several ways, for example:

$$\begin{aligned} \frac{y^6}{y^2 + 1} &= \frac{y^4(y^2 + 1) - y^4}{y^2 + 1} = y^4 - \frac{y^4}{y^2 + 1} = y^4 - \frac{y^2(y^2 + 1) - y^2}{y^2 + 1} = y^4 - y^2 + \frac{y^2}{y^2 + 1} = \\ &= y^4 - y^2 + \frac{y^2 + 1 - 1}{y^2 + 1} = y^4 - y^2 + 1 - \frac{1}{y^2 + 1} \end{aligned}$$

$$\text{or: } \frac{y^6}{y^2 + 1} = \frac{(y^6 + 1) - 1}{y^2 + 1} = \frac{(y^2 + 1)(y^4 - y^2 + 1) - 1}{y^2 + 1} = y^4 - y^2 + 1 - \frac{1}{y^2 + 1}$$

or polynomial division can also be applied.

$$\begin{aligned} \text{So } I &= \int \frac{y^6}{y^2 + 1} \, dy = \int \left(y^4 - y^2 + 1 - \frac{1}{y^2 + 1} \right) dy = \frac{y^5}{5} - \frac{y^3}{3} + y - \arctan y + c = \\ &= \frac{\tan^5 x}{5} - \frac{\tan^3 x}{3} + \tan x - x + c \end{aligned}$$

Example 2*. $I = \int \frac{1}{(x^2 + 1)^2} \, dx = ?$

Substitution: $x = x(t) = \tan t \implies t = \arctan x$

$$x'(t) = \frac{dx}{dt} = \frac{1}{\cos^2 t} \implies dx = \frac{1}{\cos^2 t} \, dt$$

$$I = \int \frac{1}{(x^2 + 1)^2} \, dx = \int \frac{1}{(\tan^2 t + 1)^2} \cdot \frac{1}{\cos^2 t} \, dt = \int \cos^4 t \cdot \frac{1}{\cos^2 t} \, dt = \int \cos^2 t \, dt =$$

$$\tan^2 t + 1 = \frac{1}{\cos^2 t}$$

$$= \int \frac{1 + \cos 2t}{2} \, dt = \frac{1}{2} t + \frac{1}{4} \sin 2t + c = \frac{1}{2} \arctan x + \frac{1}{2} \frac{x}{1 + x^2} + c$$

$$\sin 2t = 2 \sin t \cos t = 2 \tan t \cos^2 t = \frac{2 \tan t}{1 + \tan^2 t} = \frac{2x}{1 + x^2}$$

Example 3. $I = \int \frac{2x}{\sqrt{1 - x^4}} \, dx = ?$

Substitution: $t = x^2 \implies x = x(t) = \sqrt{t} = t^{\frac{1}{2}}$

$$x'(t) = \frac{dx}{dt} = \frac{1}{2} t^{-\frac{1}{2}} = \frac{1}{2\sqrt{t}} \implies dx = \frac{1}{2\sqrt{t}} \, dt$$

$$I = \int \frac{2x}{\sqrt{1 - x^4}} \, dx = \int \frac{2\sqrt{t}}{\sqrt{1 - t^2}} \cdot \frac{1}{2\sqrt{t}} \, dt = \int \frac{1}{\sqrt{1 - t^2}} \, dt = \arcsin t + c = \arcsin(x^2) + c$$

Example 4. $I = \int \frac{1}{x\sqrt{x^2 - 1}} \, dx = ?$

$$\text{Substitution: } t = \sqrt{x^2 - 1} \Rightarrow t^2 = x^2 - 1 \Rightarrow t^2 + 1 = x^2$$

$$x = x(t) = \sqrt{t^2 + 1} = (t^2 + 1)^{\frac{1}{2}}$$

$$x'(t) = \frac{dx}{dt} = \frac{1}{2} (t^2 + 1)^{-\frac{1}{2}} \cdot 2t = \frac{t}{\sqrt{t^2 + 1}} \Rightarrow dx = \frac{t}{\sqrt{t^2 + 1}} dt$$

$$\begin{aligned} I &= \int_x \frac{1}{\sqrt{x^2 - 1}} dx = \int \frac{1}{\sqrt{t^2 + 1} \cdot t} \cdot \frac{t}{\sqrt{t^2 + 1}} dt = \int \frac{1}{t^2 + 1} dt = \\ &= \operatorname{arctg} t + c = \operatorname{arctg}(\sqrt{x^2 - 1}) + c \end{aligned}$$