Calculus 1, 14th lecture

Limits of real functions

Definitions

A **function** $f : A \longrightarrow B$ is a mapping that assigns exactly one element of B to every element from A.
The set A is called the **domain** of f (notation: D_f or $\mathrm{Dom}(f)$) and
the set $f(A) = \{f(x) : x \in A\}$ is called the **range** of f (notation: R_f or $\mathrm{Ran}(f)$).

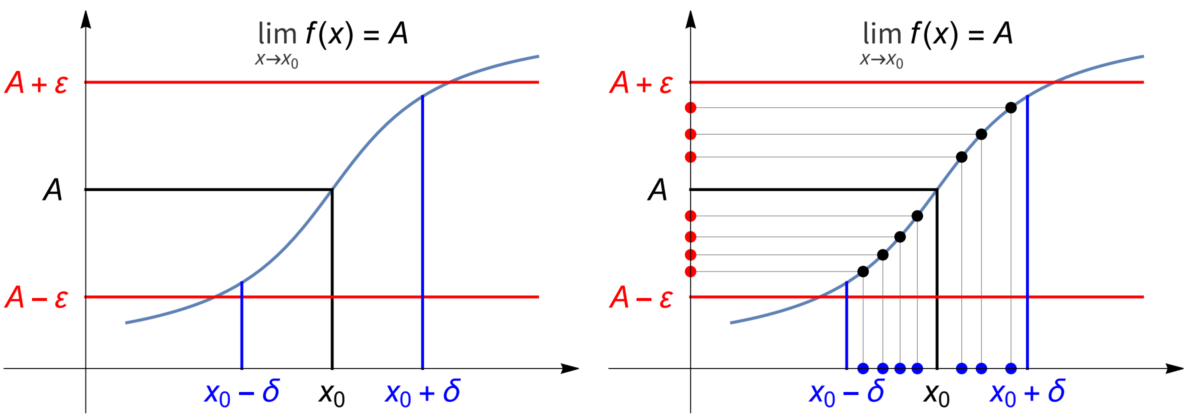
A function $f : A \longrightarrow B$ is **one-to one** or **injective** if for all $x, y \in A$: $(f(x) = f(y) \implies x = y)$.
A function $f : A \longrightarrow B$ is **onto** or **surjective** if $f(A) = B$.
A function f is **bijective** if it is injective and surjective.

The function $f : D_f \subset \mathbb{R} \longrightarrow \mathbb{R}$ is
- even, if $\forall x \in D_f$, $-x \in D_f$ and $f(x) = f(-x)$ (for example, $f(x) = x^2$ or $f(x) = \cos x$)
- odd, if $\forall x \in D_f$, $-x \in D_f$ and $f(-x) = -f(x)$ (for example, $f(x) = x^3$ or $f(x) = \sin x$)
- monotonically increasing if $\forall x, y \in D_f$ $(x < y \implies f(x) \le f(y))$
- monotonically decreasing if $\forall x, y \in D_f$ $(x < y \implies f(x) \ge f(y))$
- strictly monotonically increasing if $\forall x, y \in D_f$ $(x < y \implies f(x) < f(y))$ (for example, $f(x) = \sqrt{x}$, $f(x) = x^3$)
- strictly monotonically decreasing if $\forall x, y \in D_f$ $(x < y \implies f(x) > f(y))$
- periodic with period $p > 0$ if $\forall x \in D_f$, $x + p \in D_f$ and $f(x) = f(x + p)$ (for example, $f(x) = \sin x$)

Limit at a finite point

Definition. The limit of the function $f : D_f \subset \mathbb{R} \longrightarrow \mathbb{R}$ at the point $x_0 \in \mathbb{R}$ is $A \in \mathbb{R}$ if
(1) x_0 is a limit point of D_f $(x \in D_f')$
(2) for all $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that
if $x \in D_f$ and $0 < |x - x_0| < \delta(\varepsilon)$ then $|f(x) - A| < \varepsilon$
Notation: $\lim_{x \to x_0} f(x) = A$

Remark: $0 < |x - x_0| < \delta$ means that $x_0 - \delta < x < x_0$ or $x_0 < x < x_0 + \delta$.



One-sided limits:

Notation. The $\begin{cases}\text{right hand limit}\\ \text{left hand limit}\end{cases}$ of f at x_0 is denoted as $\begin{cases}\lim\limits_{x\to x_0+} f(x) = \lim\limits_{x\to x_0+0} f(x) = f(x_0+0)\\ \lim\limits_{x\to x_0-} f(x) = \lim\limits_{x\to x_0-0} f(x) = f(x_0-0)\end{cases}$.

Definition. Suppose $x_0 \in \mathbb{R}$ is a limit point of $\begin{cases} D_f \cap [x_0, \infty)\\ D_f \cap (-\infty, x_0]\end{cases}$. Then

$\begin{cases}\lim\limits_{x\to x_0+} f(x) = A\\ \lim\limits_{x\to x_0-} f(x) = A\end{cases}$ if for all $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that if $x \in D_f$ and $\begin{cases} x_0 < x < x_0 + \delta(\varepsilon)\\ x_0 - \delta(\varepsilon) < x < x_0\end{cases}$

then $|f(x) - A| < \varepsilon$.

Consequence. If x_0 is a limit point of D_f then $\lim\limits_{x\to x_0} f(x)$ exists if and only if $\lim\limits_{x\to x_0+} f(x)$ and $\lim\limits_{x\to x_0-} f(x)$ exist

and $\lim\limits_{x\to x_0+} f(x) = \lim\limits_{x\to x_0-} f(x)$.

Definition. Let $f : X \longrightarrow Y$ be a function and $A \subset X$. The **restriction of *f* to *A*** is the function

$f|_A : A \longrightarrow Y, \quad f|_A(x) = f(x)$.

Remark. $\lim\limits_{x\to x_0+} f(x) = \lim\limits_{x\to x_0} f|_{D_f \cap [x_0,\infty)}(x)$ and $\lim\limits_{x\to x_0-} f(x) = \lim\limits_{x\to x_0} f|_{D_f \cap (-\infty, x_0]}(x)$

Example 1. Using the definition, show that $\lim\limits_{x\to -2} \dfrac{8-2x^2}{x+2} = 8$.

Solution. We have to show that if x is "close" to x_0, that is, $|x - x_0|$ is "small", then $f(x)$ is "close" to A, that is, $|f(x) - A|$ is also "small". That is, we have to show that for all $\varepsilon > 0$ there exists $\delta > 0$ such that if $0 < |x - x_0| < \delta$, then $|f(x) - A| < \varepsilon$.

Here $x_0 = -2$. If $\varepsilon > 0$ then

$$|f(x) - A| = \left|\frac{8-2x^2}{x+2} - 8\right| = \left|\frac{2\cdot(4-x^2)}{x+2} - 8\right| = |2\cdot(2-x) - 8| =$$

$$= |-2x - 4| = |(-2)(x+2)| = 2|x+2| = 2|x-(-2)| < \varepsilon, \text{ if } |x+2| < \frac{\varepsilon}{2}$$

$\Longrightarrow$ with the choice $\delta = \delta(\varepsilon) = \dfrac{\varepsilon}{2}$ the definition holds. Remark: $-2 \notin D_f$.

For example if $\varepsilon = 10^{-2}$ then $\delta = 5\cdot 10^{-3}$.

Example 2. Using the definition, show that $\lim\limits_{x\to -3} \sqrt{1-5x} = 4$.

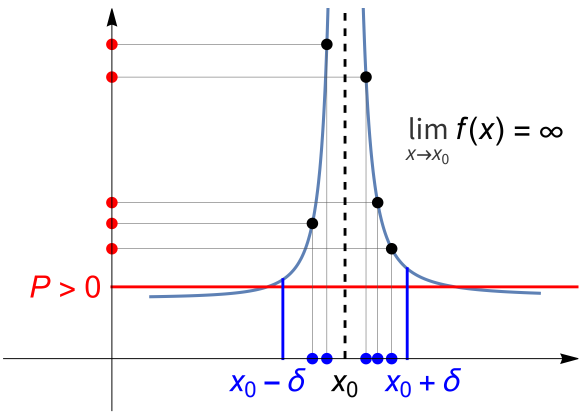
Solution. Let $\varepsilon > 0$. Then

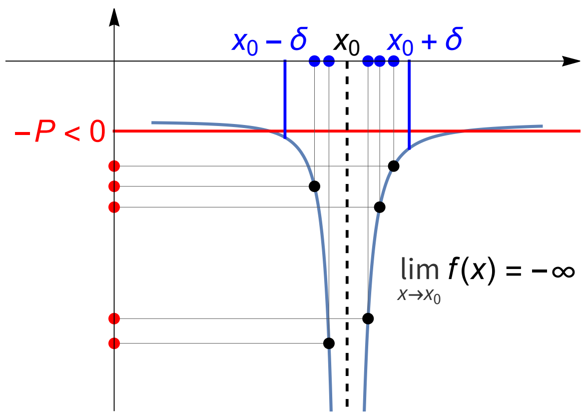
$$|f(x) - A| = \left|\sqrt{1-5x} - 4\right| = \left|\frac{1-5x-16}{\sqrt{1-5x}+4}\right| = \frac{5|x-(-3)|}{\mathbf{\sqrt{1-5x}}+4} \le \frac{5|x+3|}{\mathbf{0}+4} < \varepsilon,$$

if $|x+3| < \dfrac{4\varepsilon}{5} \Longrightarrow$ with the choice $\delta(\varepsilon) = \dfrac{4\varepsilon}{5}$ the definition holds.

Definition. Suppose $f : D_f \subset \mathbb{R} \longrightarrow \mathbb{R}$ is a function and $x_0 \in D_f{}'$. Then $\lim\limits_{x \to x_0} f(x) = \begin{cases} \infty \\ -\infty \end{cases}$ if

for all $P > 0$ there exists $\delta(P) > 0$ such that if $x \in D_f$ and $0 < |x - x_0| < \delta(P)$ then $\begin{cases} f(x) > P \\ f(x) < -P \end{cases}$.





Remark. The one-sided limits can be defined similarly:

- $\lim\limits_{x \to x_0+} f(x) = \begin{cases} \infty \\ -\infty \end{cases}$ if $\forall\, P > 0\ \exists\, \delta(P) > 0$ such that if $x \in D_f$ and $x_0 < x < x_0 + \delta(P)$ then $\begin{cases} f(x) > P \\ f(x) < -P \end{cases}$.
- $\lim\limits_{x \to x_0-} f(x) = \begin{cases} \infty \\ -\infty \end{cases}$ if $\forall\, P > 0\ \exists\, \delta(P) > 0$ such that if $x \in D_f$ and $x_0 - \delta(P) < x < x_0$ then $\begin{cases} f(x) > P \\ f(x) < -P \end{cases}$.

Example 3. $\lim\limits_{x \to 2} \dfrac{1}{(x-2)^2} = \infty$, since if $P > 0$, then $f(x) = \dfrac{1}{(x-2)^2} > P \iff 0 < |x - 2| < \dfrac{1}{\sqrt{P}}$

$\implies$ with the choice $\delta(P) = \dfrac{1}{\sqrt{P}}$ the definition holds.

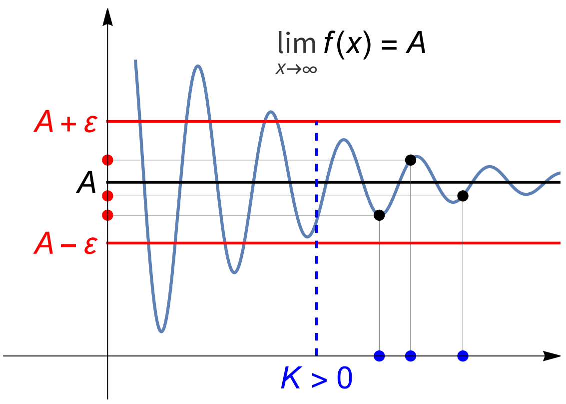
Limit at ∞ and $-\infty$

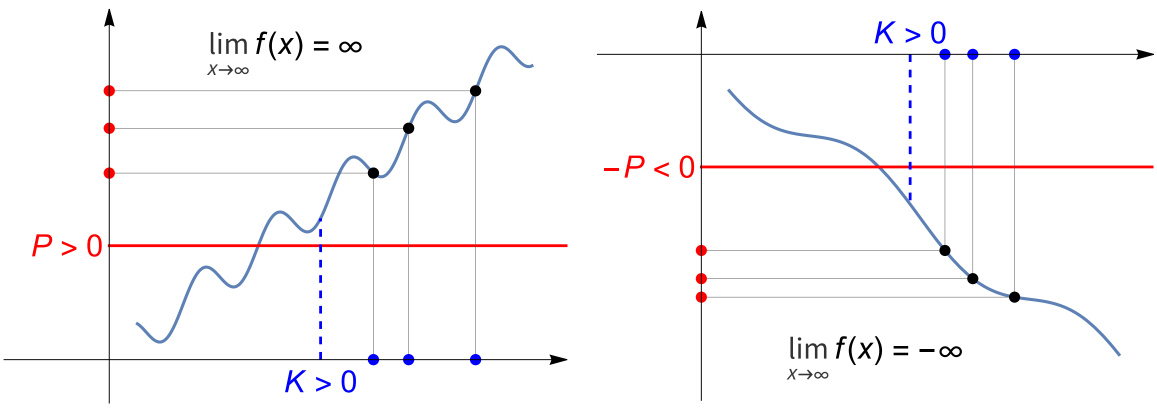
Definitions. Assume that D_f is not bounded above.

(1) $\lim\limits_{x \to \infty} f(x) = A \in \mathbb{R}$ if for all $\varepsilon > 0$ there exists $K(\varepsilon) > 0$ such that if $x > K(\varepsilon)$ then $|f(x) - A| < \varepsilon$.

(2) $\lim\limits_{x \to \infty} f(x) = \infty$ if for all $P > 0$ there exists $K(P) > 0$ such that if $x > K(P)$ then $f(x) > P$.

(3) $\lim\limits_{x \to \infty} f(x) = -\infty$ if for all $P > 0$ there exists $K(P) > 0$ such that if $x > K(P)$ then $f(x) < -P$.





Remark. If f is a sequence, that is, $D_f = \mathbb{N}^+$, then the only accumulation point of D_f is ∞, so we can investigate the limit only here.

Definitions. Assume that D_f is not bounded below.

(1) $\lim_{x\to-\infty} f(x) = A \in \mathbb{R}$ if for all $\varepsilon > 0$ there exists $K(\varepsilon) > 0$ such that if $x < -K(\varepsilon)$ then $|f(x) - A| < \varepsilon$.

(2) $\lim_{x\to-\infty} f(x) = \infty$ if for all $P > 0$ there exists $K(P) > 0$ such that if $x < -K(P)$ then $f(x) > P$.

(3) $\lim_{x\to-\infty} f(x) = -\infty$ if for all $P > 0$ there exists $K(P) > 0$ such that if $x < -K(P)$ then $f(x) < -P$.

Summary

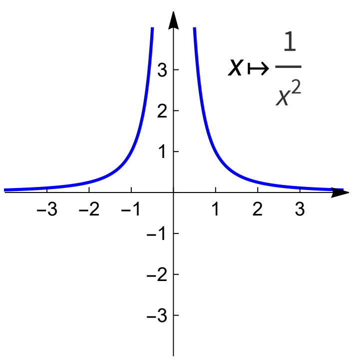
The above definitions of the limit can be summarized as follows.

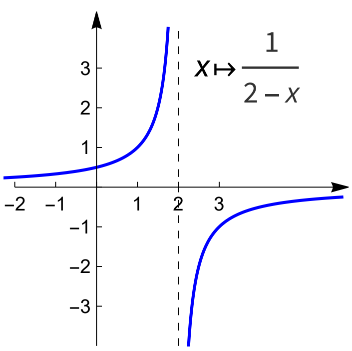
Theorem. Assume that $a \in \overline{\mathbb{R}}$ is a limit point of D_f and $b \in \overline{\mathbb{R}}$. Then $\lim_{x\to a} f(x) = b$ if and only if

for any neighbourhood J of b there exists a neighbourhood I of a such that if $x \in I \cap D_f$ and $x \neq a$ then $f(x) \in J$.

Examples

- $\lim_{x\to 0-0} \frac{1}{x^2} = \lim_{x\to 0+0} \frac{1}{x^2} = +\infty \implies \lim_{x\to 0} \frac{1}{x^2} = +\infty$
- $\lim_{x\to\infty} \frac{1}{x^2} = \lim_{x\to-\infty} \frac{1}{x^2} = 0$
- $\lim_{x\to 2-0} \frac{1}{2-x} = +\infty, \lim_{x\to 2+0} \frac{1}{2-x} = -\infty \implies \lim_{x\to 2} \frac{1}{2-x}$ doesn't exist
- $\lim_{x\to\infty} \frac{1}{2-x} = \lim_{x\to-\infty} \frac{1}{2-x} = 0$

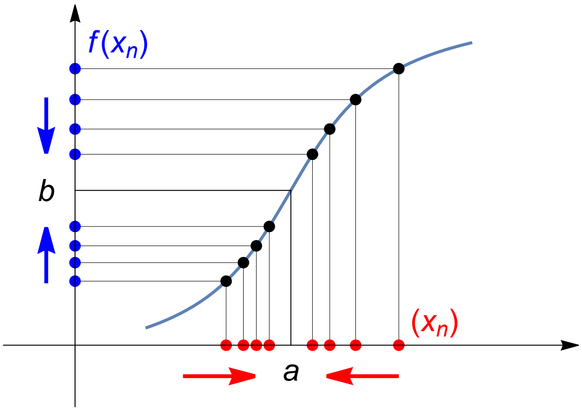




The sequential criterion for the limit of a function

In the syllabus it is called transference principle.

Theorem. Suppose $f : D_f \subset \mathbb{R} \longrightarrow \mathbb{R}$ is a function, $a, b \in \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$, and $a \in D_f'$.
Then the following two statements are equivalent.
(1) $\lim\limits_{x \to a} f(x) = b$
(2) For all sequences $(x_n) \subset D_f \setminus \{a\}$ for which $x_n \longrightarrow a$, $\lim\limits_{n \to \infty} f(x_n) = b$.



Proof. We prove it for $a, b \in \mathbb{R}$.
(1) $\Longrightarrow$ (2): • Assume that for all $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that if $0 < |x - a| < \delta(\varepsilon)$ then $|f(x) - b| < \varepsilon$.
• Let (x_n) be a sequence for which $x_n \in D_f \setminus \{a\}$ for all $n \in \mathbb{N}$ and $x_n \longrightarrow a$.
• Then for $\delta(\varepsilon) > 0$ there exists a threshold index $N(\delta(\varepsilon)) \in \mathbb{N}$ such that if $n > N(\delta(\varepsilon))$ then $|x_n - a| < \delta(\varepsilon)$.
• Thus for all $n > N(\delta(\varepsilon))$, $|f(x_n) - b| < \varepsilon$ also holds, so $f(x_n) \longrightarrow b$.

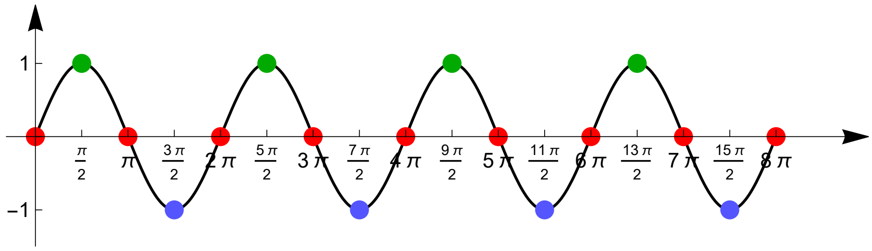
(2) $\Longrightarrow$ (1): • Indirectly, assume that (2) holds but $\lim\limits_{x \to a} f(x) \neq b$, that is,
there exists $\varepsilon > 0$ such that for all $\delta > 0$ there exists $x \in D_f$ for which $0 < |x - a| < \delta$ and $|f(x) - b| \geq \varepsilon$.
• Let $\delta_n = \frac{1}{n} > 0$ for all $n \in \mathbb{N}^+$. Then for δ_n there exists $x_n \in D_f$ such that
$$0 < |x_n - a| < \delta_n = \frac{1}{n} \text{ and } |f(x_n) - b| \geq \varepsilon.$$
• It means that $x_n \longrightarrow a$, but $\lim\limits_{n \to \infty} f(x_n) \neq b$, which is a contradiction, so $\lim\limits_{x \to a} f(x) = b$.

Remark. The theorem is useful for problems where we prove that the limit doesn't exist.

Examples

1. Show that the limit $\lim\limits_{x \to \infty} \sin(x)$ does not exist.

Solution. We give two different sequences tending to infinity such that the sequence of the corresponding function values have different limits. For example:

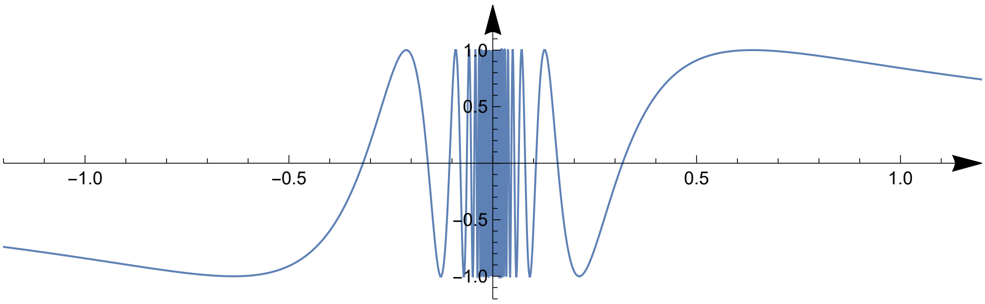


1) If $a_n = \frac{\pi}{2} + n \cdot 2\pi$, then $a_n \longrightarrow \infty$ and $\sin(a_n) = 1 \longrightarrow 1$.

2) If $b_n = n \cdot \pi$, then $b_n \longrightarrow \infty$ and $\sin(b_n) = 0 \longrightarrow 0$.

3) If $c_n = \frac{3\pi}{2} + n \cdot 2\pi$, then $c_n \longrightarrow \infty$ and $\sin(c_n) = -1 \longrightarrow -1$. $\Longrightarrow$ $\lim_{x\to\infty} \sin(x)$ doesn't exist.

2. Let $f(x) = \sin\left(\frac{1}{x}\right)$, $D_f = \mathbb{R} \setminus \{0\}$. Show that f does not have a limit at 0.



Example. Let $x_n = \frac{1}{n\pi} \longrightarrow 0$ and $y_n = \frac{1}{\frac{\pi}{2} + 2n\pi} \longrightarrow 0$. Then

- $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} \sin\left(\frac{1}{x_n}\right) = \lim_{x\to\infty} \sin(n\pi) = 0$ and
- $\lim_{n\to\infty} f(y_n) = \lim_{x\to\infty} \sin\left(\frac{1}{y_n}\right) = \lim_{x\to\infty} \sin\left(\frac{\pi}{2} + 2n\pi\right) = 1 \neq 0$ $\Longrightarrow$ $\lim_{x\to 0} \sin\left(\frac{1}{x}\right)$ doesn't exist.

Consequences

Theorem. Suppose $x_0 \in \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$ is a limit point of $D_f \cap D_g$ and $\lim_{x\to x_0} f(x) = A \in \mathbb{R}$,

$\lim_{x\to x_0} g(x) = B \in \mathbb{R}$, $c \in \mathbb{R}$. Then

(1) $\lim_{x\to x_0} (c\, f)(x) = c \cdot A$

(2) $\lim_{x\to x_0} (f \pm g)(x) = A \pm B$

(3) $\lim_{x\to x_0} (f \cdot g)(x) = A \cdot B$

(4) $\lim_{x\to x_0} \left(\frac{f}{g}\right)(x) = \frac{A}{B}$ if $B \neq 0$

(5) If $\lim_{x\to x_0} f(x) = 0$ and g is bounded in a neighbourhood of x_0 then $\lim_{x\to x_0} (f\, g)(x) = 0$.

Remark. The statements (1)-(4) are also true if $A, B \in \overline{\mathbb{R}}$ and the corresponding operations are defined in $\overline{\mathbb{R}}$.

Theorem. Suppose $x_0 \in \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$ is a limit point of $D_f \cap D_g$ and

$$\lim_{x \to x_0} f(x) = A \in \overline{\mathbb{R}}, \ \lim_{x \to x_0} g(x) = B \in \overline{\mathbb{R}}.$$

If $f(x) \leq g(x)$ for all $x \in D_f \cap D_g$ then $A \leq B$.

Theorem (Sandwich theorem for limits). Suppose that

(1) $x_0 \in \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$ is a limit point of $D_f \cap D_g \cap D_h$,

(2) $f(x) \leq g(x) \leq h(x)$ for all x in a neighbourhood of x_0 and

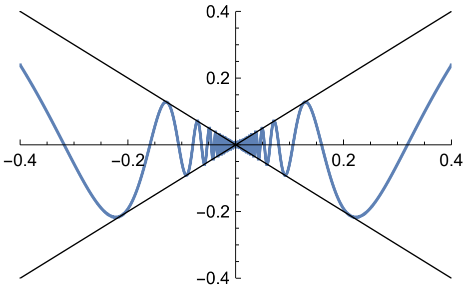
(3) $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} h(x) = b \in \overline{\mathbb{R}}$.

Then $\lim_{x \to x_0} g(x) = b$.

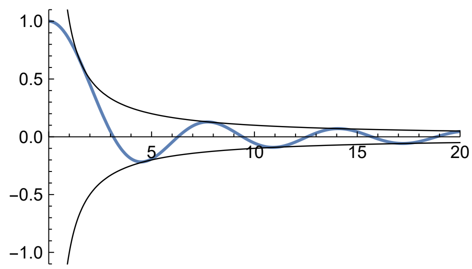
Remark. The theorem is also true for one-sided limits and if $b = \pm\infty$ then only one estimation is enough.

Example. Show that a) $\lim_{x \to 0} x \sin\left(\frac{1}{x}\right) = 0$ and b) $\lim_{x \to \infty} \frac{1}{x} \sin(x) = 0$.

a)



b)



Solution.

a) $\lim_{x \to 0} x \sin\left(\frac{1}{x}\right) = 0$, since $-\left|x\right| \leq x \sin\left(\frac{1}{x}\right) \leq \left|x\right|$, and $\lim_{x \to 0} (\,|x|\,) = \lim_{x \to 0} (-|x|) = 0$

Or: $x \longrightarrow 0$ and $\sin\left(\frac{1}{x}\right)$ is bounded, so the product also tends to 0.

b) $\lim_{x \to \infty} \frac{\sin(x)}{x} = 0$, since $-\frac{1}{x} \leq \frac{\sin(x)}{x} \leq \frac{1}{x}$ if $x > 0$, and $\lim_{x \to \infty} \left(-\frac{1}{x}\right) = \lim_{x \to \infty} \left(\frac{1}{x}\right) = 0$.

Or: $\frac{1}{x} \overset{x \to \infty}{\longrightarrow} 0$ and $\sin(x)$ is bounded, so the product also tends to 0.

Example

Theorem. $\lim_{x \to 0} \frac{\sin x}{x} = 1$

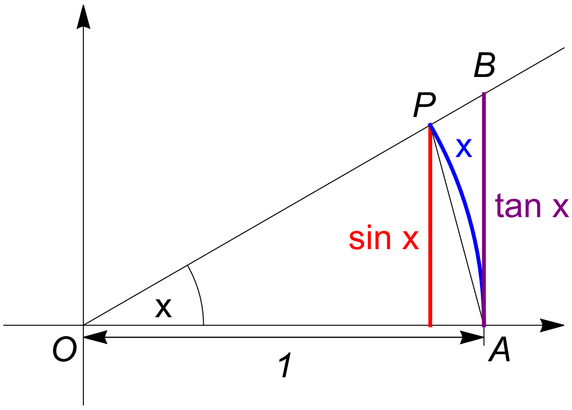
Proof. Since $f(x) = \frac{\sin x}{x}$ is even, it is enough to consider the right-hand limit $\lim_{x \to 0+} \frac{\sin x}{x}$.

Let $0 < x < \frac{\pi}{2}$.

The area of the POA triangle is $T_1 = \frac{1 \cdot \sin x}{2}$.

The area of the POA circular sector is $T_2 = \frac{1^2 \cdot x}{2}$.

The area of the OAB triangle is $T_3 = \frac{1 \cdot \tan x}{2}$.



Obviously $T_1 < T_2 < T_3 \implies \frac{1 \cdot \sin x}{2} < \frac{1^2 \cdot x}{2} < \frac{1 \cdot \tan x}{2}$.

Multiplying both sides by $\frac{2}{\sin x} > 0$: $1 < \frac{x}{\sin x} < \frac{1}{\cos x}$.

Since $\lim\limits_{x \to 0+} \frac{1}{\cos x} = 1$ then $\lim\limits_{x \to 0+} \frac{x}{\sin x} = 1 \implies \lim\limits_{x \to 0+} \frac{\sin x}{x} = 1 = \lim\limits_{x \to 0-} \frac{\sin x}{x}$

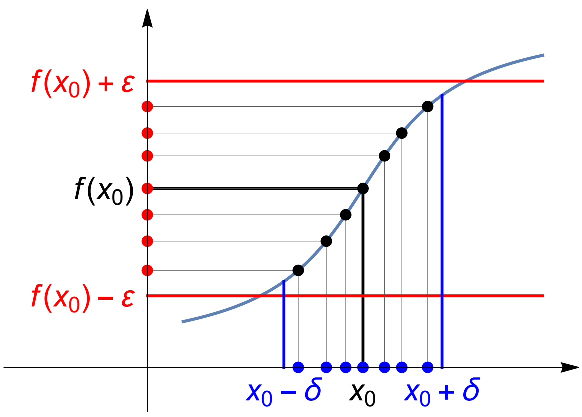
Remark. If $0 < x < \frac{\pi}{2}$, then $\sin x < x \implies |\sin x| \leq |x| \quad \forall x \in \mathbb{R}$.

Continuity

Definition. The function $f : D_f \subset \mathbb{R} \longrightarrow \mathbb{R}$ is $\begin{cases} \text{continuous} \\ \text{continuous from the left} \\ \text{continuous from the right} \end{cases}$ at the point $x_0 \in D_f$ if

for all $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that if $x \in D_f$ and $\begin{cases} |x - x_0| < \delta(\varepsilon) \\ x_0 - \delta(\varepsilon) < x \leq x_0 \\ x_0 \leq x < x_0 + \delta(\varepsilon) \end{cases}$

then $|f(x) - f(x_0)| < \varepsilon$.



Remarks. 1) f is continuous at $x_0 \in D_f \iff$ for all $\varepsilon > 0$ there exists $\delta > 0$ such that
if $x \in (B(x_0, \delta) \cap D_f$ then $f(x) \in B(f(x_0), \varepsilon)$.

2) f is $\begin{cases} \text{continuous from the right} \\ \text{continuous from the left} \end{cases}$ at $x_0 \in D_f \iff \begin{cases} f \mid_{D_f \cap [x_0, \infty)} \\ f \mid_{D_f \cap (-\infty, x_0]} \end{cases}$ is continuous at x_0.

3) f is continuous at $x_0 \in D_f \iff f$ is continuous at x_0 from the right and from the left.

Theorem. Suppose $f : D_f \subset \mathbb{R} \longrightarrow \mathbb{R}$ and $x_0 \in D_f \cap D_f'$. Then f is continuous at x_0 if and only if $\lim_{x \to x_0} f(x)$ exists and $\lim_{x \to x_0} f(x) = f(x_0)$.

Definition. f is continuous if f is continuous for all $x \in D_f$.

Notation. If $A \subset \mathbb{R}$ then $C(A, \mathbb{R})$ or $C(A)$ denotes the set of continuous functions $f : A \longrightarrow \mathbb{R}$.
For example, $f \in C([a, b])$ means that $f : [a, b] \longrightarrow \mathbb{R}$ is continuous.

The sequential criterion for continuity

Theorem: The function $f : D_f \subset \mathbb{R} \longrightarrow \mathbb{R}$ is continuous at $x_0 \in D_f$ if and only if for all sequences $(x_n) \subset D_f$ for which $x_n \longrightarrow x_0$, $\lim_{n \to \infty} f(x_n) = f(x_0)$.

Consequences

Theorem. If f and g are continuous at $x_0 \in D_f \cap D_g$ then $c\,f$, $f \pm g$ and $f\,g$ is continuous at x_0 $(c \in \mathbb{R})$.
If $g(x_0) \neq 0$ then $\frac{f}{g}$ is also continuous at x_0.

Theorem (Sandwich theorem for continuity): Suppose that
(1) there exists $\delta > 0$ such that $I = (x_0 - \delta, x_0 + \delta) \subset D_f \cap D_g \cap D_h$
(2) f and h are continuous at x_0
(3) $f(x_0) = h(x_0)$
(4) $f(x) \leq g(x) \leq h(x)$ for all $x \in I$
Then g is continuous at x_0.

Definition. The composition of the functions f and g is $(f \circ g)(x) = f(g(x))$ whose domain is
$D_{f \circ g} = \{x \in D_g : g(x) \in D_f\}$.

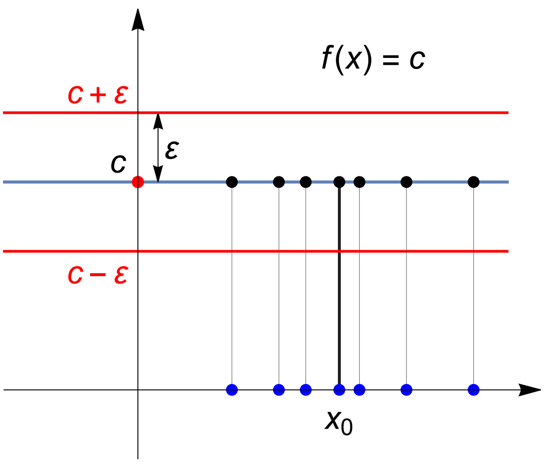
Theorem. If g is continuous at $x_0 \in D_g$ and f is continuous at $g(x_0) \in D_f$ then $f \circ g$ is continuous at x_0.

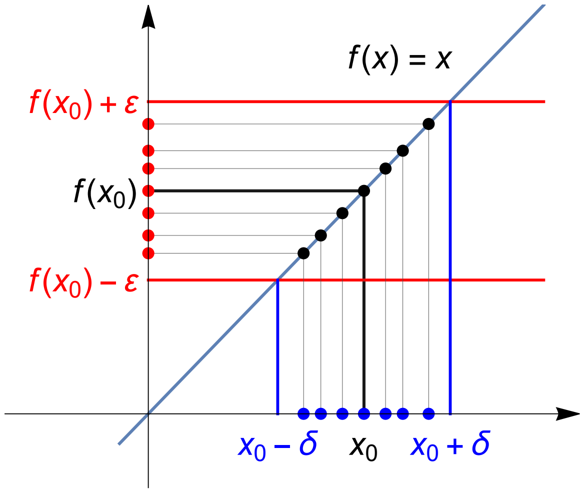
Theorem (Limit of a composition). Let a be a limit point of $D_{f \circ g}$ for which $\lim_{x \to a} g(x) = b$.
Assume that
(1) $b \in D_f$, f is continuous at b and $f(b) = c$ **or**
(2) $b \in D_f' \setminus D_f$ and $\lim_{x \to b} f(x) = c$ **or**
(3) g is injective, $b \in D_f'$ and $\lim_{x \to b} f(x) = c$.
Then $\lim_{x \to a} (f \circ g)(x) = c$.

Examples

1. Show that the constant function $f:\mathbb{R}\longrightarrow\mathbb{R},\ f(x)=c$ is continuous for all $x_0\in\mathbb{R}$.

Solution. Let $\varepsilon>0$, then with any $\delta>0$ if $|x-x_0|<\delta$, then $|f(x)-f(x_0)|=|c-c|=0<\varepsilon$.





2. Show that the function $f:\mathbb{R}\longrightarrow\mathbb{R},\ f(x)=x$ is continuous for all $x_0\in\mathbb{R}$.

Solution. Let $\varepsilon>0$, then with $\delta(\varepsilon)=\varepsilon$ if $|x-x_0|<\delta(\varepsilon)=\varepsilon$, then $|f(x)-f(x_0)|=|x-x_0|<\varepsilon$.

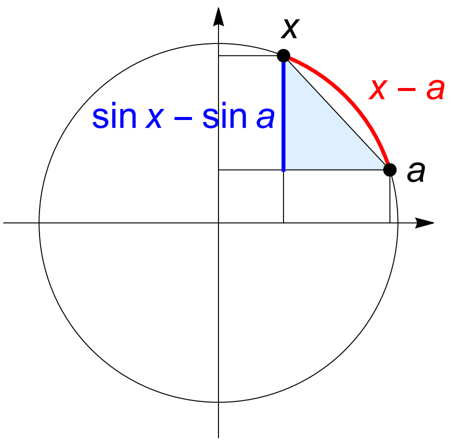
3. $f:\mathbb{R}\longrightarrow\mathbb{R},\ f(x)=x^n$ is continuous for all $x_0\in\mathbb{R}$, $n\in\mathbb{N}$, since
$f(x)=x^n=x\cdot x\cdot\ldots\cdot x\longrightarrow x_0\cdot x_0\cdot\ldots\cdot x_0=x_0^n=f(x_0)$

4. Polynomials $(P_n(x)=a_n x^n+a_{n-1}x^{n-1}+\ldots+a_1 x+a_0,\ \ a_i\in\mathbb{R})$ are continuous for all $x_0\in\mathbb{R}$.

5. $f(x)=\sin x$ and $g(x)=\cos x$ are continuous for all $x\in\mathbb{R}$.

Proof. We show that $f(x)=\sin x$ is continuous at $a\in\mathbb{R}$. Let $x\in\mathbb{R},\ x\neq a$ and consider the right-angled triangle with vertices $(\cos a,\ \sin a)$, $(\cos x,\ \sin x)$, $(\cos x,\ \sin a)$. Then the lengths of the legs are less than the length of the hypotenuse, which is less then the arc length $x-a$, that is, $|\sin x-\sin a|\le|x-a|$.

If $\varepsilon>0$ and $\delta=\varepsilon$ then for all $x\in\mathbb{R}$ for which $|x-a|<\delta$ we have that $|f(x)-f(a)|=|\sin x-\sin a|\le|x-a|<\varepsilon$, so f is continuous at a.



6. Investigate the continuity of the following functions:

a) the sign function or signum function: $\operatorname{sgn}(x) = \begin{cases} 1, & \text{ha } x > 0 \\ 0, & \text{ha } x = 0 \\ -1, & \text{ha } x < 0 \end{cases}$

b) the floor function: $f(x) = [x]$, where $[x] = \max\{k \in \mathbb{Z} : k \leq x\}$

c) the fractional part function: $f(x) = \{x\} = x - [x]$

Solution. a) $\lim\limits_{x \to 0+} \operatorname{sgn}(x) = 1 \neq \operatorname{sgn}(0) = 0 \implies f(x) = \operatorname{sgn}(x)$ is not continuous at 0 from the right

(and similarly not continuous at 0 from the left) $\implies f$ is not continuous at 0.

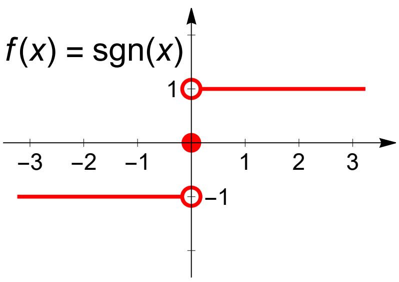
If $x \neq 0$ then f is continuous at x.

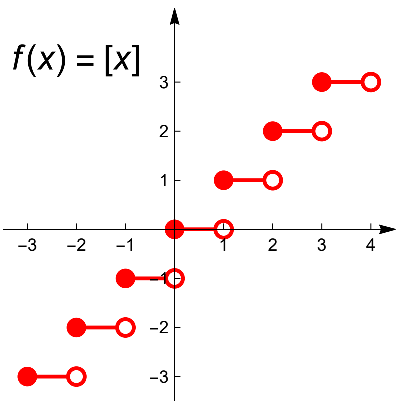
b) If $k \in \mathbb{Z}$ then $\lim\limits_{x \to k-0} [x] = k - 1$, $\lim\limits_{x \to k+0} [x] = k = [k]$

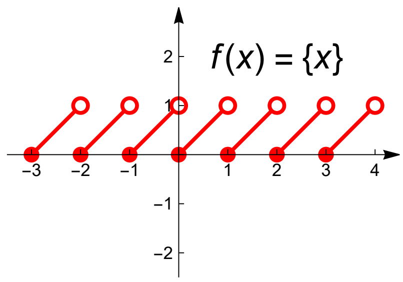
$\implies f(x) = [x]$ is continuous at k from the right but not from the left.

c) If $k \in \mathbb{Z}$ then $\lim\limits_{x \to k-0} \{x\} = 1$, $\lim\limits_{x \to k+0} \{x\} = \{k\} = 0$

$\implies f(x) = \{x\}$ is continuous at k from the right but not from the left.







7. $f(x) = \begin{cases} x \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ is continuous for all $x \in \mathbb{R}$.

8. Show that the **Dirichlet function** $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$ is not continuous at any $x \in \mathbb{R}$.

Solution.
- If $x_0 \in \mathbb{Q}$, then let $x_n \in \mathbb{R} \setminus \mathbb{Q}$ $\forall n$ such that $x_n \longrightarrow x_0$. Then $f(x_n) = 0 \longrightarrow 0 \neq 1 = f(x_0)$.
- If $x_0 \in \mathbb{R} \setminus \mathbb{Q}$, then let $x_n \in \mathbb{Q}$ $\forall n$ such that $x_n \longrightarrow x_0$. Then $f(x_n) = 1 \longrightarrow 1 \neq 0 = f(x_0)$.

9. Show an example for a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ that is continuous only at one point.

Solution. Let $f(x) = \begin{cases} x, & \text{ha } x \in \mathbb{Q} \\ -x, & \text{ha } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$. Then f is continuous only at 0

.

Since $f(x) = |x|$ for all $x \in \mathbb{R}$, then

$x_n \longrightarrow 0 \iff |x_n| \longrightarrow 0 \iff |f(x_n)| \longrightarrow 0 \iff f(x_n) \longrightarrow 0.$

Similar examples: $f(x) = \begin{cases} x, & \text{ha } x \in \mathbb{Q} \\ 0, & \text{ha } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$, $f(x) = \begin{cases} x, & \text{ha } x \in \mathbb{Q} \\ 2x, & \text{ha } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$ etc.

Types of discontinuities

Definition. We say that the function f is **discontinuous** at $x_0 \in \mathbb{R}$ or f has a discontinuity at $x_0 \in \mathbb{R}$ if x_0 is a limit point of D_f and f is not continuous at x_0.

Classification of discontinuities:

1) Discontinuity of the first kind:

a) f has a **removable discontinuity** at x_0 if $\exists \lim\limits_{x \to x_0} f(x) \in \mathbb{R}$ but $\lim\limits_{x \to x_0} f(x) \neq f(x_0)$ or $f(x_0)$ is not defined.

b) f has a **jump discontinuity** at x_0 if $\exists \lim\limits_{x \to x_0-} f(x) \in \mathbb{R}$ and $\exists \lim\limits_{x \to x_0+} f(x) \in \mathbb{R}$ but $\lim\limits_{x \to x_0-} f(x) \neq \lim\limits_{x \to x_0+} f(x)$.

2) Discontinuity of the second kind:

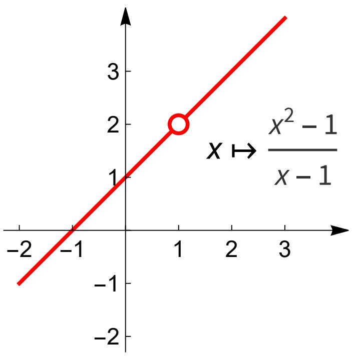
f has an **essential discontinuity** or a discontinuity of the second kind at x_0 if f has a discontinuity at x_0 but not of the first kind.

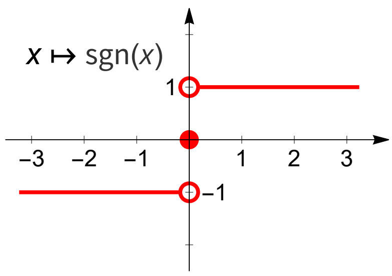
Remarks: 1. In the case of a discontinuity of the first kind, both one-sided limits exist and are finite.
2. In the case of an essential discontinuity, at least one of the one-sided limits doesn't exist or exists but is not finite.

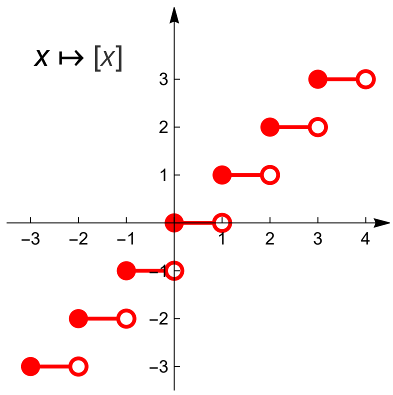
Examples

1. Discontinuity of the first kind

a) $f(x) = \dfrac{x^2 - 1}{x - 1}$ has a removable discontinuity at $x_0 = 1$.





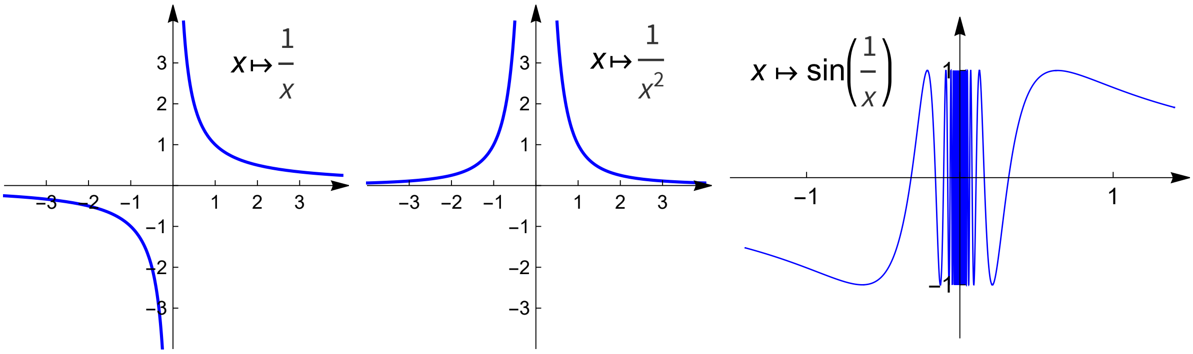


b) $f(x) = \mathrm{sgn}(x)$ has a jump discontinuity at $x = 0$.

c) $f(x) = [x]$ has a jump discontinuity for all $x \in \mathbb{Z}$.

2. Discontinuity of the second kind

a) $f_1(x) = \dfrac{1}{x}$, $f_2(x) = \dfrac{1}{x^2}$ and $f_3 = \sin \dfrac{1}{x}$ have an essential discontinuity at $x = 0$.



b) The Dirichlet function has essential discontinuities for all $x \in \mathbb{R}$.

c) The function $f(x) = e^{\frac{1}{x}}$ has an essential discontinuity at $x = 0$.

- If $x \longrightarrow 0+$, then $\frac{1}{x} \longrightarrow \infty$, and since $\lim\limits_{x \to \infty} e^x = \infty$, then $\lim\limits_{x \to 0+0} e^{\frac{1}{x}} = \infty$.
- If $x \longrightarrow 0-$, then $\frac{1}{x} \longrightarrow -\infty$, and since $\lim\limits_{x \to -\infty} e^x = 0$, then $\lim\limits_{x \to 0-0} e^{\frac{1}{x}} = 0$.