Calculus 1, 12th and 13th lecture

Basic topological concepts

Open and closed sets

Definition. The set $B(x, r) := \{y \in \mathbb{R} : |x - y| < r\} = (x - r, x + r)$ is called an open ball with center x and radius $r > 0$. This interval is also called an open neighbourhood of x with radius r.

Definitions. The set $A \subset \mathbb{R}$ is
(1) **open** if for all $x \in A$ there exists $r > 0$ such that $B(x, r) \subset A$.
(2) **closed** if its complement $\mathbb{R} \setminus A$ is open.
(3) **bounded** if there exists $r > 0$ and $x \in \mathbb{R}$ such that $A \subset B(x, r)$.

Examples. (1) $(0, 1)$ is open, $[0, 1]$ is closed, $(0, 1]$ is not open and not closed
(2) $\mathbb{Q}$ is not open and not closed
(3) The empty set $\emptyset$ and $\mathbb{R}$ are both open and closed (and they are the only such sets)
$\mathbb{R}$ is open, since it contains all open balls $\Longrightarrow \mathbb{R} \setminus \mathbb{R} = \emptyset$ is closed.
$\emptyset$ is open, since it does not contain any points $\Longrightarrow \mathbb{R} \setminus \emptyset = \mathbb{R}$ is closed.

Intersection and union

Theorem. (1) The intersection of any finite collection of open subsets of $\mathbb{R}$ is open.
(2) The union of arbitrarily many open subsets of $\mathbb{R}$ is open.

Proof. (1) Suppose $A_1, A_2, \ldots, A_n$ are open sets and let $x \in \bigcap_{i=1}^{n} A_i$.

Then for all $i = 1, \ldots, n$ there exists $r_i > 0$ such that $B(x, r_i) \subset A_i$.

If $R = \min\{r_i : i = 1, \ldots, n\}$ then $R > 0$ and $B(x, R) \subset \bigcap_{i=1}^{n} A_i$, so $\bigcap_{i=1}^{n} A_i$ is open.

(2) Suppose $\{A_i : i \in I\}$ is a collection of open sets, indexed by I.
If $x \in \bigcup_{i \in I} A_i$ then $x \in A_k$ for some $k \in I$. Since A_k is open, there exists $r > 0$,
such that $B(x, r) \subset A_k \subset \bigcup_{i \in I} A_i$, so $\bigcup_{i \in I} A_i$ is open.

Theorem.
(1) The union of any finite collection of closed subsets of $\mathbb{R}$ is closed.
(2) The intersection of arbitrarily many closed subsets of $\mathbb{R}$ is closed.

Proof. (1) Suppose $\bigcup_{i=1}^{n} A_i$ is a finite union of closed sets. Then $\mathbb{R} \setminus \bigcup_{i=1}^{n} A_i = \bigcap_{i=1}^{n} (\mathbb{R} \setminus A_i)$.

The complement of $\bigcup_{i=1}^{n} A_i$ is finite intersection of open sets, so it is open,

and therefore $\bigcup_{i=1}^{n} A_i$ is closed.

(2) Suppose $\{A_i : i \in I\}$ is a collection of closed sets, indexed by I. Then $\mathbb{R} \setminus \bigcap_{i \in I} A_i = \bigcup_{i \in I} (\mathbb{R} \setminus A_i)$.

The complement of $\bigcap_{i \in I} A_i$ is a union of a collection of open sets, so it is open,

and therefore $\bigcap_{i \in I} A_i$ is closed.

Remarks. (1) An infinite intersection of open sets is not necessarily open.

For example, $A_n = \left(-\frac{1}{n}, \frac{1}{n}\right)$ are open but $\bigcap_{n=1}^{\infty} A_n = \{0\}$ is closed.

(2) An infinite union of closed sets is not necessarily closed.

For example, $A_n = \left[-1 + \frac{1}{n}, 1 - \frac{1}{n}\right]$ are closed but $\bigcup_{n=1}^{\infty} A_n = (-1, 1)$ is open.

Examples. (1) If $x \in \mathbb{R}$, then $\{x\} \subset \mathbb{R}$ is closed, since $\mathbb{R} \setminus \{x\}$ is the union of two open intervals.

(2) $\mathbb{Z}$ is closed, since $\mathbb{R} \setminus \mathbb{Z} = \bigcup_{n=1}^{\infty} ((-n-1, -n) \cup (n-1, n))$ is a union of open sets, so

$\mathbb{R} \setminus \mathbb{Z}$ is open.

Interior, exterior and boundary points

Definition. Let $A \subset \mathbb{R}$ and $x \in \mathbb{R}$. Then

(1) x is an **interior point** of A, if there exists $r > 0$ such that $B(x, r) \subset A$.
The set of interior points of A is denoted by $\operatorname{int} A$.

(2) x is an **exterior point** of A, if there exists $r > 0$ such that $B(x, r) \cap A = \emptyset$.
The set of exterior points of A is denoted by $\operatorname{ext} A$.

(3) x is a **boundary point** of A, if for all $r > 0$: $B(x, r) \cap A \neq \emptyset$ and $B(x, r) \cap (\mathbb{R} \setminus A) \neq \emptyset$.
It means that any interval $(x - r, x + r)$ contains a point in A and a point not in A.
The set of boundary points of A is denoted by ∂A.

Remarks. (1) $\operatorname{ext} A = \operatorname{int}(\mathbb{R} \setminus A)$

(2) $\mathbb{R}$ is a disjoint union of $\operatorname{int} A$, ∂A and $\operatorname{ext} A$.

(3) $\operatorname{int} A$ and $\operatorname{ext} A$ are open, ∂A is closed.

(4) $\partial A = \partial(\mathbb{R} \setminus A)$

Limit points and isolated points

Definition. Let $A \subset \mathbb{R}$ and $x \in \mathbb{R}$. Then

(1) x is a **limit point** or **accumulation point** of A, if for all $r > 0$: $(B(x, r) \setminus \{x\}) \cap A \neq \emptyset$
It means that any interval $(x - r, x + r)$ contains a point in A that is distinct from x.
The set of limit points of A is denoted by A'.

(2) x is an **isolated point** of A, if there exists $r > 0$ such that $B(x, r) \cap A = \{x\}$
It means that x is not a limit point of A.

Remarks. (1) $\text{int} A \subset A'$, that is, every interior point of A is a limit point of A.
(2) If x is a boundary point of A, then x is a limit point or an isolated point of A.

The closure of a set

Definition. The **closure** of the set $A \subset \mathbb{R}$ is $\overline{A} := \{x \in \mathbb{R} \mid \forall r > 0 : B(x, r) \cap A \neq \emptyset\}$.

Remarks. (1) $\overline{A} = \text{int} A \cup \partial A$
(2) $\overline{A} = A \cup A'$
(3) $\overline{A}$ is closed.

Exercise 1

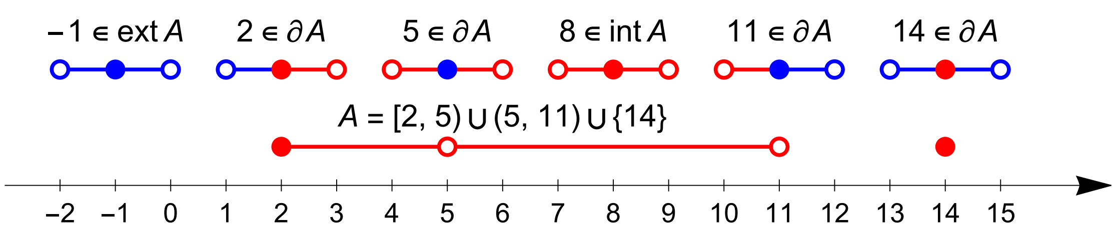
Let $A = [2, 5) \cup (5, 11) \cup \{14\}$. Find the set of interior points, boundary points, exterior points, limit points, isolated points of A and the closure of A.

Solution.

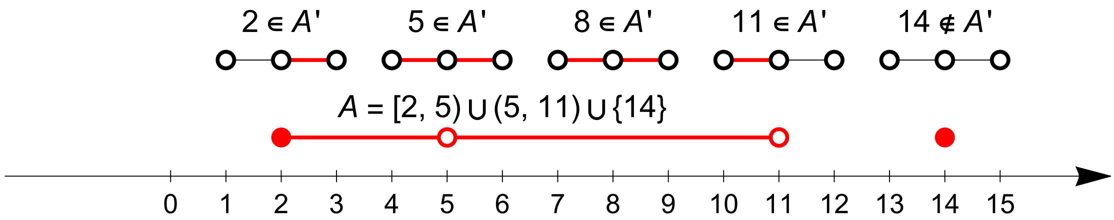
- $\text{int} A = (2, 5) \cup (5, 11)$, since these points have a neighbourhood that is a subset of A.

- $\partial A = \{2, 5, 11, 14\}$, since any neighbourhood of these points contains a point in A and a point not in A.

- $\text{ext} A = (-\infty, 2) \cup (11, 14) \cup (14, \infty)$, since these points have a neighbourhood that is disjoint from A.



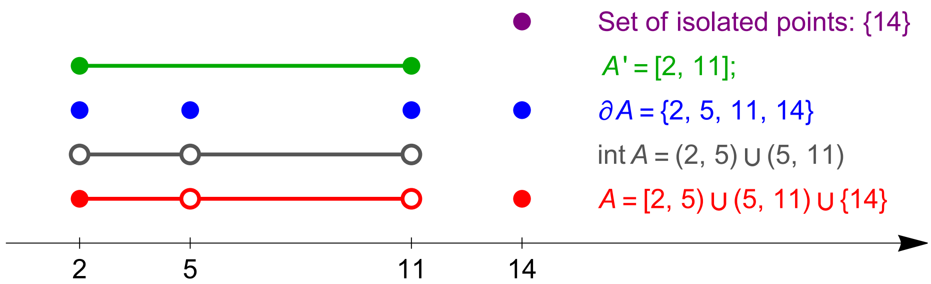
- $A' = [2, 11]$, since if $x \in A'$ then any interval $(x - r, x + r)$ contains a point in A that is distinct from x.



- The only isolated point of A is $x = 14$, since there exists an interval $(x - r, x + r)$ such that $(x - r, x + r) \cap A = \{x\}$.

- $\overline{A} = [2, 11] \cup \{14\}$, since if $x \in \overline{A}$ then any interval $(x - r, x + r)$ contains a point in A.

Let us observe that
- $\text{int}\, A \subset A'$
- If $x \in \partial A$ then $x \in A'$ of x is an isolated point of A.



Exercise 2

Let $A = \left\{\frac{1}{n} : n \in \mathbb{Z}^+\right\}$. Find the set of interior points, boundary points, limit points and isolated points of A.

Solution.

- Set of interior points: $\text{int}\, A = \emptyset$, since there is no interval that is a subset of A.

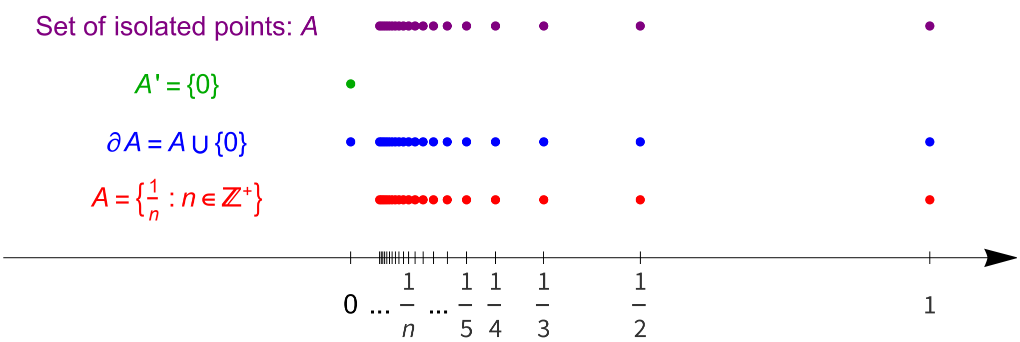
- Set of boundary points: $\partial A = A \cup \{0\}$.

  a) All points of A are boundary points, since for all $r > 0$, the interval $B\left(\frac{1}{n}, r\right) = \left(\frac{1}{n} - r, \frac{1}{n} + r\right)$ contains a point in A, that is, $\frac{1}{n}$, and a point not in A, that is, a real number that is different from the points of A.

  b) The point $0 \notin A$ is also a boundary point of A. Since for all $r > 0$ there exists $n \in \mathbb{N}$ such that $0 < \frac{1}{n} < r$, then $B(0, r)$ contains a point in A and a point not in A, say 0.

- Set of isolated points: A. All points of A are isolated points, since if $r = \frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)}$, then $B\left(\frac{1}{n}, r\right) \cap A = \left\{\frac{1}{n}\right\}$.

- Set of limit points: $A' = \{0\}$. The point $0 \notin A$ is the only limit point of A, since for all $r > 0$ there exists $n \in \mathbb{N}$ such that $0 < \frac{1}{n} < r$, so $B(0, r) \cap (A \setminus \{0\}) \neq \emptyset$.



Exercise 3

Let $A = [0, 1] \cap \mathbb{Q}$. Find the set of interior points, boundary points, limit points and isolated points of A.

Solution.
Using that any (non-empty) open interval contains both rational and irrational numbers, we get the following:
- Set of interior points: $\text{int}\,A = \emptyset$.
- Set of boundary points: $\partial A = [0, 1]$.
- Set of isolated points: $\emptyset$.
- Set of limit points: $A' = [0, 1]$.

Some examples

	Set of interior points	Set of boundary points	Set of limit points	Set of isolated points
$A = (1, 2) \cup (2, 3)$	A	$\{1, 2, 3\}$	$[1, 3]$	$\emptyset$
$A = \left\{\frac{(-1)^n}{n} : n \in \mathbb{N}\right\}$	$\emptyset$	$A \cup \{0\}$	$\{0\}$	A
$\mathbb{Z}$	$\emptyset$	$\mathbb{Z}$	$\emptyset$	$\mathbb{Z}$
$\mathbb{Q}$	$\emptyset$	$\mathbb{R}$	$\mathbb{R}$	$\emptyset$

Theorems about open and closed sets

Theorem. Let $A \subset \mathbb{R}$. Then

(1) $\operatorname{int} A$ is open;

(2) $\operatorname{int} A$ is the largest open set contained in A;

(3) $\overline{A}$ is closed;

(4) $\overline{A}$ is the smallest closed set containing A.

Consequence. Let $A \subset \mathbb{R}$. Then

(1) A is open if and only if $A = \operatorname{int} A$;

(2) A is closed if and only if $A = \overline{A}$.

Theorem. A set $A \subset \mathbb{R}$ is closed if and only if it contains all of its limit points.

Proof. a) Assume that A is closed. Then $\mathbb{R} \setminus A$ is open

$\Longrightarrow$ for all $x \in \mathbb{R} \setminus A$ there exists $r > 0$ such that $B(x, r) \subset \mathbb{R} \setminus A$

$\Longrightarrow$ if x is not in A, then x is not a limit point of A

$\Longrightarrow$ if x is a limit point of A, then x is in A $\Longrightarrow$ $A' \subset A$.

b) Assume that $A' \subset A$ and let $x \in \mathbb{R} \setminus A$. Since $x \notin A$ and $x \notin A'$ then there exists $r > 0$ such that $B(x, r) \cap A = \emptyset$

$\Longrightarrow$ for all $x \in \mathbb{R} \setminus A$ there exists $r > 0$ such that $B(x, r) \subset \mathbb{R} \setminus A$

$\Longrightarrow$ $\mathbb{R} \setminus A$ is open $\Longrightarrow$ A is closed.

Example. The set $A = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$ is not closed, since $0 \in A' \setminus A$. It is not open either, since it has no interior points.

Theorem. Let $A \subset \mathbb{R}$ be bounded. Then

(1) if $A \subset \mathbb{R}$ is closed then $\inf A$, $\sup A \in A$ (that is, A has a minimum and a maximum);

(2) if $A \subset \mathbb{R}$ is open then $\inf A$, $\sup A \notin A$.

Dense sets

Definition. Let $X, Y \subset \mathbb{R}$. Then

(1) ***X* is dense in *Y*** if $\overline{X} = Y$;

(2) X is **dense** if $\overline{X} = \mathbb{R}$.

Theorem. (1) $\mathbb{Q}$ is dense in $\mathbb{R}$;

(2) $\mathbb{R} \setminus \mathbb{Q}$ is dense in $\mathbb{R}$.

Compact sets

Definition. A set $A \subset \mathbb{R}$ is **sequentially compact** if every sequence in A has a convergent subsequence whose limit belongs to A.

Theorem (Bolzano-Weierstrass).
A set $A \subset \mathbb{R}$ is sequentially compact if and only if it is closed and bounded.

Definition. A **cover** of the set $X \subset \mathbb{R}$ is a collection of sets $C = \{A_i \subset \mathbb{R} : i \in I\}$, whose union contains X, that is, $X \subset \bigcup_{i \in I} A_i$.
An **open cover** of X is a cover such that A_i is open for every $i \in I$.
A **subcover** S of the cover C is a sub-collection $S \subset C$ that covers X, that is,

$$S = \{A_{i_k} \in C : k \in J\}, \quad X \subset \bigcup_{k \in J} A_{i_k}$$

A **finite subcover** is a subcover $\{A_{i_1}, A_{i_2}, \ldots, A_{i_n}\}$ that consists of finitely many sets.

Definition. A set $A \subset \mathbb{R}$ is **compact** if every open cover of A has a finite subcover.

Theorem (Heine-Borel or Borel-Lebesgue theorem).
A subset of $\mathbb{R}$ is compact if and only if it is closed and bounded.

Consequence. A subset of $\mathbb{R}$ is compact if and only if it is sequentially compact.

The extended set of real numbers

Definition. Let $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$ denote the extended set of real numbers. We define $-\infty \leq x \leq \infty$ for all $x \in \overline{\mathbb{R}}$. The arithmetic operations on $\mathbb{R}$ can be partially extended to $\overline{\mathbb{R}}$ as follows.

(1) $a + \infty = +\infty + a = \infty$,	$a \neq -\infty$	(5) $\dfrac{a}{\pm\infty} = 0$,	$a \in \mathbb{R}$
(2) $a - \infty = -\infty + a = -\infty$,	$a \neq +\infty$	(6) $\dfrac{\pm\infty}{a} = \pm\infty$,	$a \in (0, +\infty)$
(3) $a \cdot (\pm\infty) = \pm\infty \cdot a = \pm\infty$,	$a \in (0, +\infty]$	(7) $\dfrac{\pm\infty}{a} = \mp\infty$,	$a \in (-\infty, 0)$
(4) $a \cdot (\pm\infty) = \pm\infty \cdot a = \mp\infty$,	$a \in [-\infty, 0)$		

Definitions. The interval $(a - \varepsilon, a + \varepsilon)$ is called a neighbourhood of a if $\varepsilon > 0$.
For any $P \in \mathbb{R}$, the interval (P, ∞) is called a neighbourhood of $+\infty$ and the interval $(-\infty, P)$ is called a neighbourhood of $-\infty$.

Remark. The definition of a limit point can be extended to $\overline{\mathbb{R}}$ as follows. Let $A \subset \overline{\mathbb{R}}$ and $x \in \overline{\mathbb{R}}$. Then x is a limit point of A, if any neighbourhood of x contains a point in A that is distinct from x.

Remark. Examples for the set of limit points in $\overline{\mathbb{R}}$: $(\mathbb{N}^+)' = \{\infty\}$, $\mathbb{Z}' = \{\infty, -\infty\}$, $\mathbb{Q}' = \overline{\mathbb{R}}$, $\mathbb{R}' = \overline{\mathbb{R}}$.