# Calculus 1, 10th and 11th lecture

## Comparison test

**Theorem.** Assume that $0 \le c_n \le a_n \le b_n$ for $n > N$ where $N$ is some fixed integer. Then

(1) If $\sum_{n=1}^{\infty} b_n$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.

(2) If $\sum_{n=1}^{\infty} c_n$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

**Proof.** Denote by $s_n^a, s_n^b, s_n^c$ the $n$th partial sums of the numerical series $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$ and $\sum_{n=1}^{\infty} c_n$ respectively.

(1) **1st proof.** We use the Cauchy criterion. Let $\varepsilon > 0$ be fixed, then by the convergence of $\sum_{n=1}^{\infty} b_n$ there exists $N(\varepsilon) \in \mathbb{N}$ such that if $m > n > N(\varepsilon)$, then $\left| s_m^b - s_n^b \right| < \varepsilon$, so if $m > n > \max\{N, N(\varepsilon)\}$ then $\left| s_m^a - s_n^a \right| = \sum_{k=n+1}^{m} a_k \le \sum_{k=n+1}^{m} b_k = \left| s_m^b - s_n^b \right| < \varepsilon$, so $\sum_{n=1}^{\infty} a_n$ is convergent.

**2nd proof.** Changing finitely many terms does not affect the convergence or divergence of a series, so it may be assumed that $0 \le a_n \le b_n$ holds for all $n \in \mathbb{N}$. (If the series does not start at $n = 1$ then it can be reindexed.)

From the condition $\begin{cases} a_1 \le b_1 \\ a_2 \le b_2 \\ \dots \\ a_n \le b_n \end{cases} \implies s_n^a = a_1 + a_2 + \dots + a_n \le b_1 + b_2 + \dots + b_n = s_n^b.$

Assume that $\sum_{n=1}^{\infty} b_n$ is convergent $\implies (s_n^b)$ is bounded $\implies (s_n^a)$ is bounded

$\implies (s_n^a)$ is convergent since it is monotonically increasing $\implies \sum_{n=1}^{\infty} a_n$ is convergent.

(2) $(s_n^c)$ is monotonically increasing if $n > N$ and not bounded, so $s_n^a - s_N^a > s_n^c - s_N^c \longrightarrow \infty$ and thus $s_n^a \longrightarrow \infty$.

**Remark.** The convergence of the $p$-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ can be investigated easily with the comparison test for $p \le 1$ and $p \ge 2$.

- If $p \le 1$ then $0 < \frac{1}{n} \le \frac{1}{n^p}$ and $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent so $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is divergent.

- If $p=2$ then $\frac{1}{n^2} \le \frac{2}{n(n+1)}$ for all $n \in \mathbb{N}^+$ and $\sum_{n=1}^{\infty} \frac{2}{n(n+1)} = 2 \sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent, so $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.
- If $p>2$ then $0 < \frac{1}{n^p} \le \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent so $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent.

**Remark.** Leonhard Euler proved in 1734 that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

## Examples

**1)** Investigate the convergence of the series $\sum_{n=1}^{\infty} \frac{1}{2n+1} = \sum_{n=1}^{\infty} a_n$.

**Solution.** Here infinitely many terms are omitted from the harmonic series. By the comparison test we show that this series is still divergent.

$$a_n = \frac{1}{2n+1} > \frac{1}{2n+n} = \frac{1}{3n} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{3n} = \frac{1}{3} \sum_{n=1}^{\infty} \frac{1}{n} \quad \text{diverges} \implies \sum_{n=1}^{\infty} a_n \text{ diverges.}$$

**2)** Investigate the convergence of the series $\sum_{n=1}^{\infty} \frac{n+2}{3n^4+5} = \sum_{n=1}^{\infty} a_n$.

**Solution.** $a_n = \frac{n+2}{3n^4+5} < \frac{n+2n}{3n^4+0} = \frac{1}{n^3}$ and $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges $(p=3>1) \implies \sum_{n=1}^{\infty} a_n$ converges.

**3)** Investigate the convergence of the series $\sum_{n=1}^{\infty} \frac{2n^2-32}{n^3+8} = \sum_{n=1}^{\infty} a_n$.

**Solution.** If $n \ge 4$ then the terms of the series are positive. By the comparison test we show that the series diverges. If $n \ge 6$ then $n^2 > 32$, so

$$a_n = \frac{2n^2-32}{n^3+8} > \frac{2n^2-n^2}{n^3+8n^3} = \frac{1}{9n} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{9n} = \frac{1}{9} \sum_{n=1}^{\infty} \frac{1}{n} \quad \text{diverges} \implies \sum_{n=1}^{\infty} a_n \text{ diverges.}$$

**4)** Investigate the convergence of the series $\sum_{n=1}^{\infty} \frac{2^n+3^{n+1}}{2^{2n+3}+5} = \sum_{n=1}^{\infty} a_n$.

**Solution.** $a_n = \frac{2^n + 3\cdot 3^n}{8\cdot 4^n + 5} < \frac{3^n + 3\cdot 3^n}{8\cdot 4^n + 0} = \frac{1}{2}\left(\frac{3}{4}\right)^n$ and

$\sum_{n=1}^{\infty} \frac{1}{2}\left(\frac{3}{4}\right)^n$ is a convergent geometric series $\left(q = \frac{3}{4},\ |q| < 1\right) \implies \sum_{n=1}^{\infty} a_n$ converges.

## Error estimation for series with nonnegative terms

**Remark.** Usually we don't know the limit $s = \sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} \sum_{k=1}^{n} a_k = \lim_{n\to\infty} s_n$ but if $n$ is large then $s_n$ gives an estimation of $s$. The error for the approximation $s \approx s_n$ is $|E| = |s - s_n|$.
If $0 \le a_k \le b_k$ for $k \ge n$ then the error can be estimated with the comparison test:

$$|E| = |s - s_n| = s - s_n = \sum_{k=1}^{\infty} a_k - \sum_{k=1}^{n} a_k = \sum_{k=n+1}^{\infty} a_k \le \sum_{k=n+1}^{\infty} b_k.$$

Here $s_n \le s$, since $(s_n)$ is monotonically increasing.

**Example.** Show that the series $\sum_{n=1}^{\infty} \frac{1}{n!}$ is convergent and estimate the error if the sum of the series is approximated by the sum of the first 6 terms ($s \approx s_6$).

**Solution.** Estimate the terms from above by the terms of a convergent series:

$$\frac{1}{n!} = \frac{1}{n\cdot(n-1)\cdot(n-2)\cdot\ldots\cdot 2\cdot 1} \le \frac{1}{n(n-1)\cdot 1\cdot\ldots\cdot 1\cdot 1} = \frac{1}{n^2 - n} \le \frac{1}{n^2 - \frac{n^2}{2}} = \frac{2}{n^2}.$$

Since $\sum_{n=0}^{\infty} \frac{2}{n^2}$ converges then by the comparison test $\sum_{n=0}^{\infty} \frac{1}{n!}$ also converges.

Error estimation for the approximation $s \approx s_n$:

$$|E| = |s - s_n| = \sum_{k=n+1}^{\infty} a_k = \frac{1}{(n+1)!}\left(1 + \frac{1}{n+2} + \frac{1}{(n+2)(n+3)} + \frac{1}{(n+2)(n+3)(n+4)} + \ldots\right) \le$$

$$\le \frac{1}{(n+1)!}\left(1 + \frac{1}{n+2} + \frac{1}{(n+2)^2} + \frac{1}{(n+2)^3} + \ldots\right) = \frac{1}{(n+1)!} \sum_{k=0}^{\infty} \left(\frac{1}{n+2}\right)^k =$$

$$= \frac{1}{(n+1)!} \cdot \frac{1}{1 - \frac{1}{n+2}} = \frac{1}{(n+1)!} \cdot \frac{n+2}{n+1}$$

If $n = 6$ then $\left| s - s_n \right| \le \frac{1}{7!} \cdot \frac{8}{7} \approx 0.000226757$ and

$s_6 = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} \approx 2.718\ldots \approx e$ (here 3 digits are accurate).

## Absolute convergence

**Definition.** We say that the numerical series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent if the series $\sum_{n=1}^{\infty} |a_n|$ is convergent.

**Example.** $\sum_{n=1}^{\infty} a_1 q^{n-1}$ is absolutely convergent if $|q| < 1$.

**Theorem.** If $\sum_{n=1}^{\infty} a_n$ is absolutely convergent then it is convergent.

**Proof.** Let $\varepsilon > 0$ be fixed. If $\sum_{n=1}^{\infty} |a_n|$ is convergent then by the Cauchy criterion there exists $N \in \mathbb{N}$ such that if $m > n > N$ then $\left| |a_{n+1}| + |a_{n+2}| \ldots + |a_m| \right| < \varepsilon$. Then for all $m > n > N$

$$|s_m - s_n| = |a_{n+1} + a_{n+2} \ldots + a_m| \leq \left| |a_{n+1}| + |a_{n+2}| \ldots + |a_m| \right| < \varepsilon$$

also holds, so by the Cauchy criterion $\sum_{n=1}^{\infty} a_n$ is convergent.

**Consequence.** If $|a_n| \leq b_n$ for $n > N$ and $\sum_{n=1}^{\infty} b_n$ is convergent then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent and therefore also convergent.

**Definition.** If $\sum_{n=1}^{\infty} a_n$ is convergent but not absolutely convergent then it is **conditionally convergent**.

**Example.** $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots = \sum_{n=1}^{\infty} \left( \frac{1}{2n-1} - \frac{1}{2n} \right) = \sum_{n=1}^{\infty} \frac{1}{2n(2n-1)}$ is convergent, since $0 < \frac{1}{2n(2n-1)} \leq \frac{1}{2n \cdot n} \leq \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.

On the other hand $\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$ which is divergent, so the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ is conditionally convergent.

## Rearrangements

**Definition.** If $\pi : \mathbb{N} \longrightarrow \mathbb{N}$ is a permutation of the natural numbers (that is, every natural number appears exactly once in this sequence) then we say that $\sum_{n=1}^{\infty} a_{\pi(n)}$ is a rearrangement of $\sum_{n=1}^{\infty} a_n$.

**Theorem (Riemann rearrangement theorem).** Suppose that $\sum_{n=1}^{\infty} a_n$ is conditionally convergent and $-\infty \leq \alpha \leq \beta \leq \infty$. Then there exists a rearrangement $\sum_{n=1}^{\infty} a_n{}'$ with partial sums $s_n{}'$ such that $\liminf s_n{}' = \alpha$, $\limsup s_n{}' = \beta$.

**Theorem.** If $\sum_{n=1}^{\infty} a_n$ is absolutely convergent then every rearrangement of $\sum_{n=1}^{\infty} a_n$ converges and they all converge to the same sum.

Proof: See W. Rudin: Principles of Mathematical Analysis, page 75:
https://web.math.ucsb.edu/~agboola/teaching/2021/winter/122A/rudin.pdf

## Alternating series

**Definition.** $\sum_{n=1}^{\infty} a_n$ is an alternating series if $a_n a_{n+1} < 0$ for all $n \in \mathbb{N}$.

**Theorem (Leibniz).** Let $(a_n)$ be a monotonically decreasing sequence of positive numbers such that $a_n \xrightarrow{n\to\infty} 0$. Then the alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n = a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + \ldots$ is convergent.

**Remark.** A series with this property is called a Leibniz series.
The theorem is called the alternating series test or Leibniz's test or Leibniz criterion.

**Proof.** Since $a_n \geq a_{n+1} > 0$ for all $n \in \mathbb{N}$ then

$$\mathbf{s_{2n}} \leq s_{2n} + (a_{2n+1} - a_{2n+2}) = \mathbf{s_{2n+2}} = s_{2n+1} - a_{2n+2} \leq \mathbf{s_{2n+1}} = s_{2n-1} - (a_{2n} - a_{2n+1}) \leq \mathbf{s_{2n-1}},$$

that is, $0 \leq \mathbf{s_2} \leq \mathbf{s_4} \leq \mathbf{s_6} \leq \mathbf{s_8} \leq \ldots \leq \mathbf{s_7} \leq \mathbf{s_5} \leq \mathbf{s_3} \leq \mathbf{s_1} = a_1$.

So $(s_{2n})$ is monotonically increasing and bounded above $\Longrightarrow$ it is convergent,
and $(s_{2n+1})$ is monotonically decreasing and bounded below $\Longrightarrow$ it is convergent.

Since $s_{2n+1} - s_{2n} = a_{2n+1} \xrightarrow{n\to\infty} 0$ then $\lim_{n\to\infty} s_{2n} = \lim_{n\to\infty} s_{2n+1} = \lim_{n\to\infty} s_n \Longrightarrow$ the series is convergent.

(Or, by the Cantor axiom $\bigcap_{n=1}^{\infty} [s_{2n}, s_{2n-1}]$ is not empty and since $s_{2n-1} - s_{2n} = a_{2n} \xrightarrow{n\to\infty} 0$ then is has only one element which is the limit of $(s_n)$.)

**Error estimation:**

Let $s = \lim_{n\to\infty} s_n$. If $n$ is odd then $s_{n+1} \leq s \leq s_n$ and if $n$ is even then $s_n \leq s \leq s_{n+1}$.
In both cases the error for the approximation $s \approx s_n$ is

$$|E| = |s - s_n| \leq |s_{n+1} - s_n| = a_{n+1}.$$

## Examples

**1.** The alternating series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ is convergent, since $a_n = \frac{1}{n}$ is monotonically decreasing and $a_n \longrightarrow 0$.

**2.** Is the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{\sqrt[3]{2n+1}} = \sum_{n=1}^{\infty} (-1)^{n+1} c_n$ convergent?

**Solution.** Since $c_n = \dfrac{1}{\sqrt[3]{2n+1}}$ is monotonically decreasing and $c_n \longrightarrow 0$ then this is a Leibniz series so it is convergent.

**3.** Is the series $\sum_{n=1}^{\infty} (-1)^{n+1} \dfrac{1}{\sqrt[n]{2n+1}} = \sum_{n=1}^{\infty} (-1)^{n+1} c_n$ convergent?

**Solution.** Since $\dfrac{1}{\sqrt[n]{3} \cdot \sqrt[n]{n}} = \dfrac{1}{\sqrt[n]{2n+n}} \leq \dfrac{1}{\sqrt[n]{2n+1}} = c_n \leq \dfrac{1}{\sqrt[n]{0+1}} = 1$

and $\dfrac{1}{\sqrt[n]{3} \cdot \sqrt[n]{n}} \longrightarrow \dfrac{1}{1 \cdot 1} = 1$ then by the sandwich theorem $\lim\limits_{n\to\infty} c_n = 1$.

So $\lim\limits_{n\to\infty} (-1)^{n+1} c_n$ doesn't exist, and thus by the $n$th term test the series diverges.

**4.** Is the series $\sum_{n=1}^{\infty} (-1)^{n+1} \dfrac{n+1}{n^2+2} = \sum_{n=1}^{\infty} (-1)^{n+1} c_n$ convergent?

**Solution. 1)** $0 < c_n = \dfrac{n+1}{n^2+2} = \dfrac{\frac{1}{n} + \frac{1}{n^2}}{1 + \frac{2}{n^2}} \longrightarrow \dfrac{0+0}{1+0} = 0.$

**2)** It is not obvious that $(c_n)$ is monotonically decreasing, since both the numerator and the denominator increases.

$$c_{n+1} \leq c_n \iff \frac{(n+1)+1}{(n+1)^2+2} \leq \frac{n+1}{n^2+2}$$
$$\iff (n+2)\left(n^2+2\right) \leq (n+1)\left(n^2+2n+3\right)$$
$$\iff n^3+2n^2+2n+4 \leq n^3+n^2+2n^2+2n+3n+3$$
$$\iff 0 \leq n^2+3n-1 \quad \text{and this is true for all } n \in \mathbb{N}.$$

Since the steps are equivalent then $c_{n+1} \leq c_n$ also holds for all $n \in \mathbb{N}$, so $(c_n)$ is monotonically decreasing. Then by the Leibniz criterion the series converges.

**Remark.** If the sum of the series is approximated by $s_{100}$ then the error is

$$|E| = |s - s_{100}| \leq c_{101} = \frac{101+1}{101^2+2}.$$

## Root test (Cauchy)

**Theorem (Root test):** Assume that $a_n > 0$ and $\limsup \sqrt[n]{a_n} = R$. Then

(1) if $R < 1$, then $\sum_{n=1}^{\infty} a_n$ is convergent;

(2) if $R > 1$, then $\sum_{n=1}^{\infty} a_n$ is divergent.

**Proof.** (1) • Suppose that $R < 1$, then there exists $\varepsilon > 0$ such that $R + \varepsilon < 1$.

• By the definition of the limsup, for this $\varepsilon$ there exists $N \in \mathbb{N}$ such that if $n > N$ then $\sqrt[n]{a_n} < R + \varepsilon$, since if $\sqrt[n]{a_n} \geq R + \varepsilon$ would hold for infinitely many $n$ then this subsequence would have a limit point greater than $R$.

- Thus $a_n \leq (R+\varepsilon)^n$ if $n > N$, and since $\sum_{n=1}^{\infty} (R+\varepsilon)^n$ is a convergent geometric series then by the comparison test, $\sum_{n=1}^{\infty} a_n$ is also convergent.

(2) • Suppose that $R > 1$, then there exists $\varepsilon > 0$ and a subsequence of $\sqrt[n]{a_n}$ such that $\sqrt[n_k]{a_{n_k}} \geq R - \varepsilon > 1$.

- Then for the terms of this subsequence $a_{n_k} \geq (R-\varepsilon)^{n_k} > 1$ $\Longrightarrow \lim_{n_k \to \infty} a_{n_k} \neq 0 \Longrightarrow \lim_{n \to \infty} a_n \neq 0 \Longrightarrow$ the series is divergent by the $n$th term test.

**Consequence.** Assume $\limsup \sqrt[n]{|a_n|} = R$. Then

(1) if $R < 1$, then $\sum_{n=1}^{\infty} a_n$ is convergent, since it is absolutely convergent;

(2) if $R > 1$, then $\sum_{n=1}^{\infty} a_n$ is divergent, since if $\lim_{n\to\infty} |a_n| \neq 0$, then $\lim_{n\to\infty} a_n \neq 0$.

**Remark.** If $R = 1$ then we don't know anything about the convergence of the series, for example

1) $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent and $\sqrt[n]{\frac{1}{n}} \longrightarrow 1$

2) $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent and $\sqrt[n]{\frac{1}{n^2}} \longrightarrow 1$

## Ratio test (D'Alambert)

**Theorem (Ratio test):** Assume that $a_n > 0$. Then

(1) if $\limsup \frac{a_{n+1}}{a_n} < 1$, then $\sum_{n=1}^{\infty} a_n$ is convergent;

(2) if $\liminf \frac{a_{n+1}}{a_n} > 1$, then $\sum_{n=1}^{\infty} a_n$ is divergent.

**Proof.** (1) • Suppose that $R = \limsup \frac{a_{n+1}}{a_n} < 1$, then similarly as in the previous proof, there exists $\varepsilon > 0$ and $N \in \mathbb{N}$ such that if $n \geq N$ then $\frac{a_{n+1}}{a_n} < R + \varepsilon < 1$.

- Thus $a_{N+1} < (R+\varepsilon)\, a_N$

$$a_{N+2} < (R+\varepsilon)\, a_{N+1} < (R+\varepsilon)^2\, a_N$$

...

$$a_{n+1} < (R+\varepsilon)\, a_n = (R+\varepsilon)^{n+1-N}\, a_N = \frac{a_N}{(R+\varepsilon)^N} \cdot (R+\varepsilon)^{n+1}$$

so we can apply the comparison test similarly as in the proof of the root test.

(2) • Suppose that $\liminf \frac{a_{n+1}}{a_n} > 1$, then there exists $\varepsilon > 0$ and $N \in \mathbb{N}$ such that if $n \geq N$ then $\frac{a_{n+1}}{a_n} > R - \varepsilon > 1$.

- Since $a_n > 0$ then $a_{n+1} > a_n$, so $(a_n)$ is monotonic increasing $\Longrightarrow \lim\limits_{n\to\infty} a_n \neq 0$

  $\Longrightarrow$ the series is divergent by the $n$th term test.

**Consequence.** Assume $a_n \neq 0$ for all $n \in \mathbb{N}$. Then

(1) if $\limsup \left| \frac{a_{n+1}}{a_n} \right| < 1$, then $\sum\limits_{n=1}^{\infty} a_n$ is convergent, since it is absolutely convergent;

(2) if $\liminf \left| \frac{a_{n+1}}{a_n} \right| > 1$, then $\sum\limits_{n=1}^{\infty} a_n$ is divergent, since if $\lim\limits_{n\to\infty} |a_n| \neq 0$, then $\lim\limits_{n\to\infty} a_n \neq 0$.

**Remark.** If $\limsup \frac{a_{n+1}}{a_n} = 1$ or $\liminf \frac{a_{n+1}}{a_n} = 1$ then we don't know anything about the convergence of the series, for example

1) $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ is divergent and $\frac{a_{n+1}}{a_n} = \frac{\frac{1}{n+1}}{\frac{1}{n}} = \frac{n}{n+1} \longrightarrow 1$

2) $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ is convergent and $\frac{a_{n+1}}{a_n} = \frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} = \frac{n^2}{(n+1)^2} \longrightarrow 1$

**Remark.** The ratio test is a consequence of the root test and the following theorem.
The proof of this theorem contains a very interesting step.

**1)** Recall that

- if $x < B$ (or $x \leq B$) **for all** $B > 0$ then $x \leq 0$.

**2)** Similarly, we can prove $x \leq y$ in the following way:

- if $x \leq B$ **for all** $B > y$ then $x \leq y$.

**Theorem.** Assume that $a_n > 0$. Then $\liminf \frac{a_{n+1}}{a_n} \leq \liminf \sqrt[n]{a_n} \leq \limsup \sqrt[n]{a_n} \leq \limsup \frac{a_{n+1}}{a_n}$.

**Proof. 1)** We prove that $\limsup \sqrt[n]{a_n} \leq \limsup \frac{a_{n+1}}{a_n}$.

Let $\limsup \frac{a_{n+1}}{a_n} = C$ and let $B > C$ be an arbitrary real number.

Then by the definition of the lim sup, there exists $N \in \mathbb{N}$ such that if $k \geq N$ then $\frac{a_{k+1}}{a_k} < B$.

$\Longrightarrow a_{N+1} < B\, a_N, \quad a_{N+2} < B\, a_{N+1} < B^2 a_N, \quad \ldots$

So if $n > N$ then $a_n < B^{n-N} a_N \Longrightarrow \sqrt[n]{a_n} < \sqrt[n]{B^{n-N}}\ \sqrt[n]{a_N} = B \cdot \sqrt[n]{\frac{a_N}{B^N}}$

$\Longrightarrow \limsup \sqrt[n]{a_n} \leq \lim\limits_{n\to\infty} B \cdot \sqrt[n]{\frac{a_N}{B^N}} = B.$

We obtained that the following implication holds **for all** $B > C$:

$\limsup \frac{a_{n+1}}{a_n} < B \Longrightarrow \limsup \sqrt[n]{a_n} \leq B.$

From this it follows that $\limsup \sqrt[n]{a_n} \leq \limsup \frac{a_{n+1}}{a_n}$.

**2)** $\liminf \sqrt[n]{a_n} \le \limsup \sqrt[n]{a_n}$ is obvious.

**3)** The proof of $\liminf \frac{a_{n+1}}{a_n} \le \liminf \sqrt[n]{a_n}$ is similar to case 1).

**Consequence.** If $a_n > 0$ for all $n$ and $\exists \lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = \alpha \in \mathbb{R}$ then $\exists \lim\limits_{n\to\infty} \sqrt[n]{a_n} = \alpha$.

**Remark.** It is a consequence of the previous inequalities that the root test is "stronger" than the ratio test. Consider the series

$$\sum_{n=1}^{\infty} a_n = \frac{1}{2} + \frac{1}{3} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{2^3} + \frac{1}{3^3} + \ldots, \text{ where } a_{2k-1} = \frac{1}{2^k} \text{ and } a_{2k} = \frac{1}{3^k},\ k \ge 1.$$

With the root test: • If $n$ is odd, then $\sqrt[n]{a_n} = \sqrt[2k-1]{a_{2k-1}} = \sqrt[2k-1]{\frac{1}{2^k}} \longrightarrow \frac{1}{\sqrt{2}}$ and

• if $n$ is even, then $\sqrt[n]{a_n} = \sqrt[2k]{a_{2k}} = \sqrt[2k]{\frac{1}{3^k}} = \frac{1}{\sqrt{3}}$.

$\Longrightarrow \limsup \sqrt[n]{a_n} = \frac{1}{\sqrt{2}} < 1 \Longrightarrow$ the series is convergent.

With the ratio test: • If $n$ is even, then $\frac{a_{n+1}}{a_n} = \frac{a_{2k+1}}{a_{2k}} = \frac{\frac{1}{2^{k+1}}}{\frac{1}{3^k}} = \frac{3^k}{2^{k+1}} \longrightarrow \infty$ and

• if $n$ is odd, then $\frac{a_{n+1}}{a_n} = \frac{a_{2k}}{a_{2k-1}} = \frac{\frac{1}{3^k}}{\frac{1}{2^k}} = \frac{2^k}{3^k} \longrightarrow 0$.

$\Longrightarrow \limsup \frac{a_{n+1}}{a_n} = \infty > 1$ and $\liminf \frac{a_{n+1}}{a_n} = 0 < 1$

$\Longrightarrow$ the ratio test cannot be used here.

## Cauchy product

**Definition:** The Cauchy product of the series $\sum\limits_{n=0}^{\infty} a_n$ and $\sum\limits_{n=0}^{\infty} b_n$ is the series $\sum\limits_{n=0}^{\infty} c_n$

where

$$c_n = a_0 b_n + a_1 b_{n-1} + \ldots + a_n b_0 = \sum_{k=0}^{n} a_k b_{n-k}$$

| | $a_0$ | $a_1$ | $a_2$ | $a_3$ | ... |
|---|---|---|---|---|---|
| $b_0$ | $a_0 b_0$ | $a_1 b_0$ | $a_2 b_0$ | $a_3 b_0$ | |
| $b_1$ | $a_0 b_1$ | $a_1 b_1$ | $a_2 b_1$ | | |
| $b_2$ | $a_0 b_2$ | $a_1 b_2$ | | | |
| $b_3$ | $a_0 b_3$ | | | | |
| ... | | | | | |

## Mertens' theorem

**Theorem (Mertens).** If $\sum_{n=0}^{\infty} a_n$ is absolutely convergent and $\sum_{n=0}^{\infty} b_n$ is convergent, then their Cauchy product is convergent and its sum is $\sum_{n=0}^{\infty} c_n = \sum_{n=0}^{\infty}\sum_{k=0}^{n} a_k\, b_{n-k} = \left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{n=0}^{\infty} b_n\right)$.

**Proof.** Let $A = \sum_{n=0}^{\infty} a_n$, $\quad B = \sum_{n=0}^{\infty} b_n$,

$$A_n = \sum_{k=0}^{n} a_k, \quad B_n = \sum_{k=0}^{n} b_k, \quad C_n = \sum_{k=0}^{n} c_k = \sum_{k=0}^{n}\sum_{i=0}^{k} a_i\, b_{k-i}, \quad \beta_n = B_n - B.$$

Then

$$C_n = a_0\, b_0 + (a_0\, b_1 + a_1\, b_0) + (a_0\, b_2 + a_1\, b_1 + a_2\, b_0) + \ldots + (a_0\, b_n + a_1\, b_{n-1} + \ldots + a_n\, b_0) =$$
$$= a_0\, B_n + a_1\, B_{n-1} + a_n\, B_{n-2} + \ldots + a_n\, B_0 =$$
$$= a_0(B + \beta_n) + a_1(B + \beta_{n-1}) + a_2(B + \beta_{n-2}) + \ldots + a_n(B + \beta_0) =$$
$$= A_n\, B + \boldsymbol{(a_0\, \beta_n + a_1\, \beta_{n-1} + a_2\, \beta_{n-2} + \ldots + a_n\, \beta_0)}.$$

Let $\boldsymbol{\gamma_n = a_0\, \beta_n + a_1\, \beta_{n-1} + a_2\, \beta_{n-2} + \ldots + a_n\, \beta_0}$.

We have to show that $C_n \longrightarrow A\,B$. Since $A_n\, B \longrightarrow A\,B$, it is enough to show that $\boldsymbol{\lim_{n\to\infty} \gamma_n = 0}$.

Let $\alpha = \sum_{n=0}^{\infty} |\, a_n \,|$. (Here we use that $\sum_{n=0}^{\infty} a_n$ is absolutely convergent.) Let $\varepsilon > 0$ be given.

Since $B = \sum_{n=0}^{\infty} b_n$ then $\beta_n \longrightarrow 0$, so there exists $N \in \mathbb{N}$ such that $|\, \beta_n \,| \le \varepsilon$ if $n \ge N$. In this case

$$|\, \gamma_n \,| \le |\, \beta_0\, a_n + \ldots \beta_N\, a_{n-N} \,| + |\, \beta_{N+1}\, a_{n-N-1} + \ldots + \beta_n\, a_0 \,| \le$$
$$\le |\, \beta_0\, a_n + \ldots \beta_N\, a_{n-N} \,| + |\, \beta_{N+1} \,| \cdot |\, a_{n-N-1} \,| + \ldots + |\, \beta_n \,| \cdot |\, a_0 \,| \le$$
$$\le \left|\, \beta_0\, a_n + \ldots \beta_N\, a_{n-N} \,\right| + \varepsilon \cdot \sum_{n=0}^{n-N-1} \left|\, a_n \,\right| \le$$
$$\le |\, \beta_0\, a_n + \ldots \beta_N\, a_{n-N} \,| + \varepsilon\, \alpha.$$

If $N$ is fixed and $n \longrightarrow \infty$ then $|\, \beta_0\, a_n + \ldots \beta_N\, a_{n-N} \,| \longrightarrow 0$ since $a_k \longrightarrow \infty$ as $k \longrightarrow \infty$.
So we get that $\limsup |\, \gamma_n \,| \le \varepsilon\, \alpha$. Since $\varepsilon$ is arbitrary, it follows that $\lim_{n\to\infty} \gamma_n = 0$.

**Remark.** If both $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are absolutely convergent then their Cauchy product is also absolutely convergent.

**Theorem (Abel).** Assume that $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are two convergent series and their Cauchy product is also convergent. Then its sum is $\sum_{n=0}^{\infty} c_n = \sum_{n=0}^{\infty}\sum_{k=0}^{n} a_k\, b_{n-k} = \left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{n=0}^{\infty} b_n\right)$.

**Remark.** In general it is not true that the Cauchy-product of two convergent series is convergent.

For example let $\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} b_n = \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$. These are Leibniz series, so they are convergent.

Then $c_n = \sum_{k=0}^{n} a_k\, b_{n-k} = \sum_{k=0}^{n} \frac{(-1)^n}{\sqrt{k+1}} \frac{(-1)^{n-k}}{\sqrt{n-k+1}} = (-1)^n \sum_{k=0}^{n} \frac{1}{\sqrt{k+1} \cdot \sqrt{n-k+1}}$.

Using the AM-GM inequality $\frac{a+b}{2} \geq \sqrt{a\, b}$, we get that

$|\ c_n\ | = \sum_{k=0}^{n} \frac{1}{\sqrt{k+1} \cdot \sqrt{n-k+1}} \geq \sum_{k=0}^{n} \frac{2}{(k+1)+(n-k+1)} = \sum_{k=0}^{n} \frac{2}{n+2} = \frac{2}{n+2}(n+1)$, since the terms are independent of $k$.

Therefore $\left| c_n \right| \geq 2 \cdot \frac{n+1}{n+2} \longrightarrow 2$, so $\lim_{n\to\infty} c_n \neq 0 \implies$ the Cauchy-product is divergent.

## Examples

**Example 1.** If $|\ x\ | < 1$ then $\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \ldots = \frac{1}{1-x}$ and

$\sum_{k=0}^{\infty} (-x)^k = 1 - x + x^2 - x^3 + \ldots = \frac{1}{1+x}$.

| | 1 | $x$ | $x^2$ | $x^3$ | ... |
|---|---|---|---|---|---|
| 1 | 1 | $\boldsymbol{x}$ | $\boldsymbol{x^2}$ | $x^3$ | |
| $-x$ | $\boldsymbol{-x}$ | $\boldsymbol{-x^2}$ | $-x^3$ | | |
| $x^2$ | $\boldsymbol{x^2}$ | $x^3$ | | | |
| $-x^3$ | $-x^3$ | | | | |
| ... | | | | | |

The Cauchy-product is $\sum_{n=0}^{\infty} \sum_{k=0}^{n} x^k (-x)^{n-k} = 1 + (\boldsymbol{x - x}) + (\boldsymbol{x^2 - x^2 + x^2}) + (x^3 - x^3 + x^3 - x^3) + \ldots =$

$= 1 + 0 + x^2 + 0 + x^4 + 0 + x^6 + \ldots = \sum_{k=0}^{\infty} x^{2k} = \sum_{k=0}^{\infty} (x^2)^k = \frac{1}{1-x^2} = \frac{1}{1-x} \cdot \frac{1}{1+x} = \left(\sum_{k=0}^{\infty} x^k\right)\left(\sum_{k=0}^{\infty} (-x)^k\right)$

**Example 2.** Since $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$ if $|\ x\ | < 1$ then

$$\frac{1}{(1-x)^2} = \left(\sum_{k=0}^{\infty} x^k\right)^2 = \sum_{n=0}^{\infty} \sum_{k=0}^{n} x^k x^{n-k} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} x^n = \sum_{n=0}^{\infty} (n+1)\, x^n$$

**Example 3.** $\left(\sum_{k=0}^{\infty} \frac{1}{n!}\right)^2 = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{1}{k!\,(n-k)!} = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \frac{n!}{k!\,(n-k)!} = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} = \sum_{n=0}^{\infty} \frac{2^n}{n!}$

# Power series

**Definitions.** The series $\sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0 + a_1 (x - x_0) + a_2 (x - x_0)^2 + \ldots$ is called a **power series** with center $x_0$, where $a_n$ is the coefficient of the $n$th term.

The domain of convergence of the power series is $H = \left\{ x \in \mathbb{R} : \sum_{n=0}^{\infty} a_n (x - x_0)^n \text{ converges} \right\}$.

The **radius of convergence** of the power series is $R = \dfrac{1}{\limsup \sqrt[n]{|a_n|}}$.

**Remarks.** $H$ is not empty, since the series converges for $x = x_0$.

Since $\sqrt[n]{|a_n|} \geq 0$, then $0 \leq \limsup \sqrt[n]{|a_n|} \leq \infty$.

If $\limsup \sqrt[n]{|a_n|} = \infty$ then $R = 0$ and if $\limsup \sqrt[n]{|a_n|} = 0$ then $R = \infty$.

If $\lim\limits_{n \to \infty} \left| \dfrac{a_{n+1}}{a_n} \right|$ exists then $R = \lim\limits_{n \to \infty} \left| \dfrac{a_n}{a_{n+1}} \right|$.

**Theorem (Cauchy-Hadamard):** Denote by $R$ the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n (x - x_0)^n$. Then

(1) if $|x - x_0| < R$, then the series is absolutely convergent, and

(2) if $|x - x_0| > R$, then the series is divergent.

**Proof.** We define $\dfrac{1}{+0} = +\infty$ and $\dfrac{1}{+\infty} = 0$. By the root test

$$\limsup \sqrt[n]{|a_n| \cdot |x - x_0|^n} = |x - x_0| \cdot \limsup \sqrt[n]{|a_n|} = \frac{|x - x_0|}{R}$$

Then $\dfrac{|x - x_0|}{R} < 1 \iff |x - x_0| < R \implies$ the series is absolutely convergent

and $\dfrac{|x - x_0|}{R} > 1 \iff |x - x_0| > R \implies$ the series is divergent.

**Consequence.** (1) If $R = 0$ then for all $x \neq x_0$, $|x - x_0| > 0 = R$, so the series diverges and if $x = x_0$ then it converges. Then $H = \{x_0\}$.

(2) If $R = \infty$ then for all $x \in \mathbb{R}$, $|x - x_0| < R$, so the series is absolutely convergent. Then $H = \mathbb{R}$.

(3) If $0 < R < \infty$, then $(x_0 - R, x_0 + R) \subset H \subset [x_0 - R, x_0 + R]$ and the endpoints of the interval must be investigated separately.