Calculus 1, 8th and 9th lecture

Bolzano-Weierstrass theorem

Theorem: Every sequence has a monotonic subsequence.

Proof. First we introduce the following concept: a_k is called a **peak element** if $a_n \leq a_k$ for all $n > k$. Then two cases are possible.

Case 1: There are infinitely many peak elements. If $n_1 < n_2 < n_3 < \ldots$ are indexes for which $a_{n_1}, a_{n_2}, a_{n_3}, \ldots$ are peak elements, then the sequence $a_{n_1}, a_{n_2}, a_{n_3}, \ldots$ is monotonically decreasing.

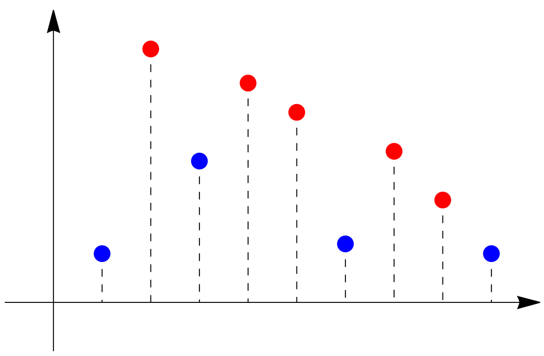
Case 2: There are finitely many peak elements (or none). It means that there exists an index n_0 such that for all $n \geq n_0$, a_n is not a peak element.

$\Longrightarrow$ Since a_{n_0} is not a peak element, there exists $n_1 > n_0$ such that $a_{n_1} > a_{n_0}$.

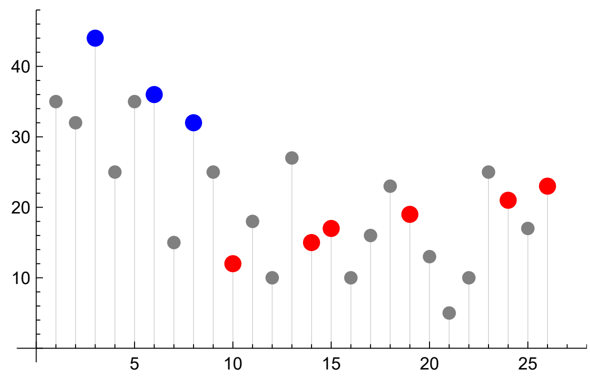
Since a_{n_1} is not a peak element, there exists $n_2 > n_1$ such that $a_{n_2} > a_{n_1}$, etc.

In this case the sequence $a_{n_0}, a_{n_1}, a_{n_2}, \ldots$ is strictly monotonic increasing.

Case 1:



Case 2:



Theorem (Bolzano-Weierstrass): Every bounded sequence has a convergent subsequence.

Proof: Because of the previous theorem there exists a monotonic subsequence and since it is bounded then it is convergent.

Remark. The Bolzano-Weierstrass theorem is not true in the set of rational numbers.

Let $(b_n) = (1, 1.4, 1.41, 1.414, \ldots) \longrightarrow \sqrt{2} \notin \mathbb{Q}$, then $b_n \in \mathbb{Q}$ and $b_n \in [1, 2]$ for all n, that is, (b_n) is bounded.

Each subsequence of (b_n) converges to $\sqrt{2}$, so (b_n) does not have a subsequence converging to a rational number.

Cauchy sequences

Definition. (a_n) is a **Cauchy sequence** if for all $\varepsilon > 0$ there exists $N(\varepsilon) \in \mathbb{N}$ such that if $n, m > N$ then $|a_n - a_m| < \varepsilon$.

Statement: If (a_n) is a Cauchy sequence, then it is bounded, since for all $\varepsilon > 0$ and $n \in \mathbb{N}$,

$$\min\{a_{N+1} - \varepsilon, a_1, \ldots, a_N\} \le a_n \le \max\{a_{N+1} + \varepsilon, a_1, \ldots, a_N\}.$$

Theorem. (a_n) is convergent if and only if it is a Cauchy sequence.

Proof. a) Let $\varepsilon > 0$ be fixed. If $\lim_{n\to\infty} a_n = A$, then for $\frac{\varepsilon}{2}$ there exists $N \in \mathbb{N}$ such that if $n > N$ then $|a_n - A| < \frac{\varepsilon}{2}$.

So if $n, m > N$ then $|a_n - a_m| = |a_n - A + A - a_m| \le |a_n - A| + |A - a_m| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

b) If (a_n) is a Cauchy sequence then it is bounded. Define $c_n = \inf\{a_n, a_{n+1}, \ldots\}$ and $d_n = \sup\{a_n, a_{n+1}, \ldots\}$.

Then $c_n \le c_{n+1} \le d_{n+1} \le d_n$, so by the Cantor-axiom $\bigcap_{n=1}^{\infty} [c_n, d_n] \ne \emptyset$.

Since for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that if $n > N$ then $|c_n - d_n| < \varepsilon$, then it means that the intersection has only one element A, which is the limit of the sequence ($|A - a_n| < \max\{|c_n - a_n|, |d_n - a_n|\} < \varepsilon$).

Remark. The theorem expresses the fact that the terms of a convergent sequence are also arbitrarily close to each other if their indexes are large enough. The theorem can be used to prove convergence even if the limit is not known.

Example. $a_n = (-1)^n$ is not convergent, since $|a_n - a_{n+1}| = |(-1)^n - (-1)^{n+1}| = 2 \ge \varepsilon$ if $\varepsilon \le 2$.

Remark. A Cauchy sequence is not necessarily convergent in the set of rational numbers.
For example $(a_n) = (1, 1.4, 1.41, 1.414, \ldots) \longrightarrow \sqrt{2} \notin \mathbb{Q}$.
(a_n) is a Cauchy sequence, since $|a_{n+k} - a_n| < 10^{-N}$ if $n > N$ and $k \in \mathbb{N}$ is arbitrary, but the limit of (a_n) is not rational.

An important example

Let $s_n = \sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$. Prove that $\lim_{n\to\infty} s_n = \infty$.

Solution. Let $\varepsilon \leq \frac{1}{2}$ and $m = 2n$. Then with

$$s_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} \quad \text{and} \quad s_m = s_{2n} = \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}\right) + \left(\frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n}\right),$$

we get that

$$|\, s_m - s_n \,| = |\, s_{2n} - s_n \,| = \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n} > \frac{1}{2n} + \frac{1}{2n} + \ldots + \frac{1}{2n} = n \cdot \frac{1}{2n} = \frac{1}{2} \geq \varepsilon,$$

so (s_n) is not a Cauchy sequence. Since (s_n) is monotonically increasing, then $s_n \longrightarrow \infty$.

Limit points or accumulation points of a sequence

Definition. For any $P \in \mathbb{R}$, the interval (P, ∞) is called a neighbourhood of $+\infty$ and the interval $(-\infty, P)$ is called a neighbourhood of $-\infty$.

Definition. $A \in \mathbb{R} \cup \{\infty, -\infty\}$ is called a **limit point** or **accumulation point** of (a_n) if any neighbourhood of A contains infinitely many terms of (a_n).
Or equivalently there exists a subsequence (a_{n_k}) such that $a_{n_k} \xrightarrow{n\to\infty} A$.

Examples

See the figures on page 1: https://math.bme.hu/~nagyi/calculus1-2022/calculus1-04-05.pdf

Sequence	**Limit points**	**Limit**	
1) $a_n = \frac{1}{n}$	$t = 0$	$\lim_{n\to\infty} a_n = 0$	$\Longrightarrow (a_n)$ converges
2) $a_n = \frac{(-1)^n}{n}$	$t = 0$	$\lim_{n\to\infty} a_n = 0$	$\Longrightarrow (a_n)$ converges
3) $a_n = (-1)^n$	$t_1 = -1, \ t_2 = 1$	$\lim_{n\to\infty} a_n$ doesn't exist	$\Longrightarrow (a_n)$ diverges
4) $a_n = n^2$	$t = +\infty$	$\lim_{n\to\infty} a_n = +\infty$	$\Longrightarrow (a_n)$ diverges
5) $a_n = \frac{n}{n+1}$	$t = 1$	$\lim_{n\to\infty} a_n = 1$	$\Longrightarrow (a_n)$ converges
6) $a_n = (-1)^n \frac{n}{n+1}$	$t_1 = -1, \ t_2 = 1$	$\lim_{n\to\infty} a_n$ doesn't exist	$\Longrightarrow (a_n)$ diverges
7) $a_n = \frac{1}{2^n}$	$t = 0$	$\lim_{n\to\infty} a_n = 0$	$\Longrightarrow (a_n)$ converges
8) $a_n = (-2)^n$	$t_1 = -\infty, \ t_2 = \infty$	$\lim_{n\to\infty} a_n$ doesn't exist	$\Longrightarrow (a_n)$ diverges

Theorem. Every sequence has at least one limit point.

Proof. We proved that every sequence has a monotonic subsequence.
If it is bounded, then it has a finite limit, so it is a limit point of the sequence.
If the subsequence is not bounded, then it tends to ∞ or $-\infty$, so ∞ or $-\infty$ is a limit point of the sequence.

Definition. • If the set of limit points of (a_n) is bounded above, then its supremum is called the **limes superior** of (a_n) (notation: $\limsup a_n$).

• If the set of limit points of (a_n) is bounded below, then its infimum is called the **limes inferior** of (a_n) (notation: $\liminf a_n$).

• If (a_n) is not bounded above, then we define $\limsup a_n = \infty$.
• If (a_n) is not bounded below, then we define $\liminf a_n = -\infty$.

Theorem. (a_n) is convergent if and only if $\limsup a_n = \liminf a_n = A \in \mathbb{R}$.

Proof. 1) If (a_n) is convergent, then all of its subsequences tend to the same limit as (a_n). Then the only element of the set of the limit points will be the limsup and the liminf of the sequence.

2) Let $\limsup a_n = \liminf a_n = A$ and let $\varepsilon > 0$ be fixed.
If we assume indirectly that $\lim_{n\to\infty} a_n \neq A$ then it means that there are infinitely many terms $n_1 < n_2 < \ldots \in \mathbb{N}$ such that $|\, a_n - A \,| \geq \varepsilon$.
Then (a_{n_k}) has a limit point which differs from A, so we arrived at a contradiction.

Examples

1. Let $a_n = 2^{(-1)^n n}$. Find $\limsup a_n$ and $\liminf a_n$.

Solution. 1) If n is even: $n = 2k$, then $(-1)^{2k} = 1 \implies a_{2k} = 2^{2k} = 4^k \longrightarrow \infty$

2) If n is odd: $n = 2k+1$, then $(-1)^{2k+1} = -1 \implies a_{2k+1} = 2^{-(2k+1)} = \dfrac{1}{2\cdot 4^k} \longrightarrow 0$

The limit points of the sequence are 0 and $\infty \implies \liminf a_n = 0$, $\limsup a_n = \infty$

2. Let $a_n = \dfrac{n^2 + n^2 \sin\left(\frac{n\pi}{2}\right)}{2n^2 + 3n + 7}$. Find the limit points of (a_n). Calculate $\limsup a_n$ and $\liminf a_n$.

Solution. $\sin\left(\dfrac{n\pi}{2}\right) = \begin{cases} 1, & \text{if } n = 1, 5, 9, \ldots \\ 0, & \text{if } n = 0, 2, 4, 6, 8, \ldots \\ -1, & \text{if } n = 3, 7, 11, \ldots \end{cases} \implies$ Depending on the value of n,

we have to investigate the behaviour of three subsequences.

1) If $n = 2k$ then $\sin\left(\dfrac{n\pi}{2}\right) = 0$, so the subsequence is $a_n = \dfrac{n^2}{2n^2+3n+7} \longrightarrow \dfrac{1}{2}$

2) If $n = 4k+1$ then $\sin\left(\dfrac{n\pi}{2}\right) = 1$, so the subsequence is $a_n = \dfrac{2n^2}{2n^2+3n+7} \longrightarrow 1$

3) If $n = 4k-1$ then $\sin\left(\dfrac{n\pi}{2}\right) = -1$, so the subsequence is $a_n = 0 \longrightarrow 0$

The limit points of the sequence are $0, \dfrac{1}{2}, 1 \implies \liminf a_n = 0$, $\limsup a_n = 1$

3. Let $a_n = \dfrac{3^{2n+1} + (-4)^n}{5 + 9^{n+1}}$ and $b_n = a_n \cdot \cos(n\,\pi)$

Find $\limsup a_n$, $\liminf a_n$, $\limsup b_n$, $\liminf b_n$.

Solution. 1) $a_n = \dfrac{3 \cdot 9^n + (-4)^n}{5 + 9 \cdot 9^n} = \dfrac{9^n}{9^n} \cdot \dfrac{3 + \left(-\frac{4}{9}\right)^n}{5 \cdot \left(\frac{1}{9}\right)^n + 9} \longrightarrow \dfrac{3+0}{0+9} = \dfrac{1}{3}$

$\Longrightarrow \lim\limits_{n\to\infty} a_n = \liminf a_n = \limsup a_n = \dfrac{1}{3}$

The sequence (a_n) is convergent, since it has only one limit point.

2) $\cos(n\,\pi) = (-1)^n \Longrightarrow$ if n is even, then $b_n = a_n \longrightarrow \dfrac{1}{3}$

if n is odd, then $b_n = -a_n \longrightarrow -\dfrac{1}{3}$

$\Longrightarrow \liminf b_n = -\dfrac{1}{3}$, $\limsup b_n = \dfrac{1}{3}$, so $\lim\limits_{n\to\infty} b_n$ does not exist.

4. Calculate the limit of the following sequences (if it exists) and find the limit superior and limit inferior.

a) $a_n = \dfrac{-4^n + 3^{n+1}}{1 + 4^n}$ b) $b_n = \dfrac{(-4)^n + 3^{n+1}}{1 + 4^n}$ c) $c_n = \dfrac{(-4)^n + 3^{n+1}}{1 + 4^{2n}}$

Solution. a) $a_n = \dfrac{-4^n + 3 \cdot 3^n}{1 + 4^n} = \dfrac{4^n}{4^n} \cdot \dfrac{-1 + 3 \cdot \left(\frac{3}{4}\right)^n}{\left(\frac{1}{4}\right)^n + 1} \longrightarrow \dfrac{-1+0}{0+1} = -1$

$\Longrightarrow \lim\limits_{n\to\infty} a_n = \liminf a_n = \limsup a_n = -1$

b) $b_n = \dfrac{(-4)^n + 3 \cdot 3^n}{1 + 4^n} = \dfrac{(-4)^n}{4^n} \cdot \dfrac{1 + 3 \cdot \left(-\frac{3}{4}\right)^n}{\left(\frac{1}{4}\right)^n + 1} = (-1)^n \cdot \beta_n$, where $\beta_n = \dfrac{1 + 3 \cdot \left(-\frac{3}{4}\right)^n}{\left(\frac{1}{4}\right)^n + 1} \longrightarrow \dfrac{1+0}{0+1} = 1$

If n is even: $b_n = \beta_n \longrightarrow 1$

If n is odd: $b_n = -\beta_n \longrightarrow -1$

$\Longrightarrow \liminf b_n = -1$, $\limsup b_n = 1$, so $\lim\limits_{n\to\infty} b_n$ does not exist.

c) $c_n = \dfrac{(-4)^n + 3 \cdot 3^n}{1 + 16^n} = \dfrac{(-4)^n}{16^n} \cdot \dfrac{1 + 3 \cdot \left(-\frac{3}{4}\right)^n}{\left(\frac{1}{16}\right)^n + 1} \longrightarrow 0 \cdot \dfrac{1+0}{0+1} = 0$

$\Longrightarrow \lim\limits_{n\to\infty} c_n = \liminf c_n = \limsup c_n = 0$

Numerical series

Definition

Definition. Suppose that (a_n) is a sequence and define the sequence of **partial sums** as

$$s_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \ldots a_n.$$

If (s_n) is convergent, then the **numerical series** $\sum_{n=1}^{\infty} a_n$ is convergent,

and its sum is $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} \sum_{k=1}^{n} a_k = \lim_{n\to\infty} s_n = s \in \mathbb{R}$.

Examples

1. a) $\sum_{k=1}^{\infty} 1 = ?$ **b)** $\sum_{k=1}^{\infty} (-1)^{k+1} = ?$

Solution. a) $\sum_{k=1}^{\infty} 1 = 1 + 1 + 1 + 1 + \ldots = \infty$

Here $s_n = \sum_{k=1}^{n} 1 = n \implies \lim_{n\to\infty} s_n = \infty \implies$ the series is divergent (and its sum is infinity).

b) $\sum_{k=1}^{\infty} (-1)^{k+1} = 1 - 1 + 1 - 1 + \ldots + (-1)^k + \ldots$

Here $s_{2k+1} = 1 \longrightarrow 1$ and $s_{2k} = 0 \longrightarrow 0$, so (s_n) has two limit points.

$\implies$ The series is divergent (and its sum doesn't exist).

2. $$\sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^k = \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{1}{2}\right)^k = \lim_{n\to\infty} \left(\frac{1}{2} + \left(\frac{1}{2}\right)^2 + \ldots + \left(\frac{1}{2}\right)^n\right) = \lim_{n\to\infty} \frac{1}{2} \cdot \frac{\left(\frac{1}{2}\right)^n - 1}{\frac{1}{2} - 1} = \frac{1}{2} \cdot \frac{0 - 1}{-\frac{1}{2}} = 1,$$

so the series is convergent.

A telescoping series

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k(k+1)} = \lim_{n\to\infty} \left(\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \ldots + \frac{1}{n(n+1)}\right) =$$

$$= \lim_{n\to\infty} \left(1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} \ldots + \frac{1}{n} - \frac{1}{n+1}\right) = \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right) = 1,$$ so the series is convergent.

The harmonic series

Theorem. The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Proof. $s_{2^n} = \sum_{k=1}^{2^n} \frac{1}{k} = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \ldots + \left(\frac{1}{2^{n-1}+1} + \ldots + \frac{1}{2^n}\right) \geq$

$$\geq 1 + \frac{1}{2} + 2 \cdot \frac{1}{4} + 4 \cdot \frac{1}{8} + \ldots + 2^{n-1} \cdot \frac{1}{2^n} = 1 + \frac{n}{2} \xrightarrow{n \to \infty} \infty, \text{ so } \lim_{n\to\infty} s_{2^n} = \infty.$$

If $n > 2^k$ then $s_n \geq s_{2^k}$, so $\lim_{n\to\infty} s_n = \infty$ and therefore $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$.

Remark. The name of the harmonic series comes from the fact that for all $n \geq 2$, a_n is the harmonic mean of a_{n-1} and a_{n+1}, that is,

$$a_n = \frac{2}{\frac{1}{a_{n-1}} + \frac{1}{a_{n+1}}} = \frac{2}{\frac{1}{\frac{1}{n-1}} + \frac{1}{\frac{1}{n+1}}} = \frac{2}{(n-1)+(n+1)} = \frac{1}{n}.$$

The divergence of the series is very slow, for example

$$\sum_{n=1}^{100} \frac{1}{n} \approx 5.18738, \quad \sum_{n=1}^{10^4} \frac{1}{n} \approx 9.78761, \quad \sum_{n=1}^{10^5} \frac{1}{n} \approx 12.0901, \quad \sum_{n=1}^{10^6} \frac{1}{n} \approx 14.3927$$

Remark. If a finite number of terms in a series are omitted or changed then the fact of convergence or divergence doesn't change. However, the sum of a convergent series changes.

The geometric series

Theorem. $1 + q + q^2 + \ldots = \sum_{n=0}^{\infty} q^n = \frac{1}{1-q}$ if $|q| < 1$ and the series is divergent otherwise.

Proof. If $a_n = q^n$ then $s_n = \sum_{k=1}^{n} a_k = \sum_{k=0}^{n} q^k = \begin{cases} \frac{q^{n+1}-1}{q-1} & \text{if } q \neq 1 \\ n+1 & \text{if } q = 1 \end{cases}$

1) If $q = 1$ then $\lim_{n\to\infty} s_n = \infty$.

2) If $q > 1$ then $\lim_{n\to\infty} s_n = \infty$, since $\lim_{n\to\infty} q^{n+1} = \infty$.

3) If $-1 < q < 1$ then $\lim_{n\to\infty} s_n = \frac{1}{1-q}$, since $\lim_{n\to\infty} q^{n+1} = 0$.

4) If $q \leq -1$ then $\lim_{n\to\infty} s_n$ does not exist, since $\lim_{n\to\infty} q^n$ does not exist.

Similarly, $\sum_{n=0}^{\infty} a \cdot q^n = \frac{a}{1-q}$, $\sum_{n=k}^{\infty} a \cdot q^n = \frac{a \cdot q^k}{1-q}$ if $|q| < 1$. $\left(\text{sum} = \frac{\text{first term}}{1 - \text{ratio}}\right)$

Sum and constant multiple

Theorem: Assume $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent, $\sum_{n=1}^{\infty} d_n$ is divergent, and $c \in \mathbb{R} \setminus \{0\}$. Then

(1) $\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$

(2) $\sum_{n=1}^{\infty} c\, a_n = c \sum_{n=1}^{\infty} a_n$

(3) $\sum_{n=1}^{\infty} (a_n + d_n)$ is divergent

(4) $\sum_{n=1}^{\infty} c\, d_n$ is divergent

Proof. All statements follow from the properties of the sequences.

Example. $\sum_{k=2}^{\infty} \frac{3^{k+1} + 5(-2)^{k+3}}{4^k} = ?$

Solution. $\sum_{k=2}^{\infty} \frac{3^{k+1} + 5(-2)^{k+3}}{4^k} = \sum_{k=2}^{n} \frac{3 \cdot 3^k - 5 \cdot 8 \cdot (-2)^k}{4^k} = 3 \cdot \sum_{n=2}^{\infty} \left(\frac{3}{4}\right)^k - 40 \cdot \sum_{k=2}^{\infty} \left(-\frac{2}{4}\right)^k =$

$$= 3 \cdot \frac{\left(\frac{3}{4}\right)^2}{1 - \frac{3}{4}} - 40 \cdot \frac{\left(-\frac{1}{2}\right)^2}{1 - \left(-\frac{1}{2}\right)} = \frac{1}{12}$$

The series is the sum of two convergent geometric series.

Cauchy criterion

Theorem: The numerical series $\sum_{n=1}^{\infty} a_n$ converges if and only if for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$

such that if $m > n > N$ then $|s_m - s_n| = \sum_{k=n+1}^{m} a_k = |a_{n+1} + a_{n+2} + \ldots + a_m| < \varepsilon$.

Proof: It is trivially true, since the Cauchy criterion for number sequences can be applied for (s_n).

Example. Is the series $\sum_{k=1}^{\infty} (-1)^{n+1} \frac{1}{n}$ convergent or divergent? (alternating harmonic series)

Solution. The series is convergent. Let $m > n$ and $m = n + k$. Then

$$|s_m - s_n| = |s_{n+k} - s_n| = |a_{n+1} + a_{n+2} + \ldots + a_{n+k}| = \left| \frac{(-1)^{n+2}}{n+1} + \frac{(-1)^{n+3}}{n+2} + \frac{(-1)^{n+4}}{n+3} + \ldots + \frac{(-1)^{n+k+1}}{n+k} \right| =$$

$$= \left| \frac{1}{n+1} - \frac{1}{n+2} + \frac{1}{n+3} - \ldots + \frac{(-1)^{k+1}}{n+k} \right|.$$

Using that $\frac{1}{n+1}-\frac{1}{n+2}>0$, $\frac{1}{n+2}-\frac{1}{n+3}>0$ etc. we get the following.

1) If k is even then

$$|\, s_{n+k}-s_n\,| = \left(\frac{1}{n+1}-\frac{1}{n+2}\right)+\left(\frac{1}{n+3}-\frac{1}{n+4}\right)+\ldots+\left(\frac{1}{n+k-1}-\frac{1}{n+k}\right)=$$
$$=\frac{1}{n+1}-\left(\frac{1}{n+2}-\frac{1}{n+3}\right)-\ldots-\left(\frac{1}{n+k}\right)<\frac{1}{n+1}$$

2) If k is odd then

$$|\, s_{n+k}-s_n\,| = \left(\frac{1}{n+1}-\frac{1}{n+2}\right)+\left(\frac{1}{n+3}-\frac{1}{n+4}\right)+\ldots+\left(\frac{1}{n+k-2}-\frac{1}{n+k-1}\right)+\frac{1}{n+k}=$$
$$=\frac{1}{n+1}-\left(\frac{1}{n+2}-\frac{1}{n+3}\right)-\ldots-\left(\frac{1}{n+k-1}-\frac{1}{n+k}\right)<\frac{1}{n+1}.$$

Then $\left|\, s_{n+k}-s_n\,\right| < \frac{1}{n+1}<\varepsilon$ if $n>\frac{1}{\varepsilon}-1$, so with the choice $N(\varepsilon)\geq\left[\frac{1}{\varepsilon}-1\right]$ the statement holds.

Later we will see that this is a Leibniz series, so it is convergent.

The *n*th term test

Theorem: If $\sum_{n=1}^{\infty} a_n$ is convergent then $\lim_{n\to\infty} a_n = 0$.

1st proof: Apply the Cauchy criterion with the choice $m = n+1$. Then

$$|\, s_{n+1}-s_n\,| = |\, a_{n+1}\,| < \varepsilon \text{ if } n > N(\varepsilon), \text{ so } \lim_{n\to\infty} a_n = 0.$$

2nd proof: Let $\lim_{n\to\infty} s_n = s\in\mathbb{R}$, then $s_n = s_{n-1}+a_n \implies a_n = s_n - s_{n-1}\longrightarrow s-s=0$.

Remark. The theorem can also be stated in the following form:

If $\lim_{n\to\infty} a_n \neq 0$ or if the limit doesn't exist then $\sum_{n=1}^{\infty} a_n$ diverges.

Remark. The condition $\lim_{n\to\infty} a_n = 0$ is necessary but not sufficient for the convergence of $\sum_{n=1}^{\infty} a_n$.

For example, the harmonic series $\sum_{n=1}^{\infty}\frac{1}{n}$ is divergent but $\lim_{n\to\infty}\frac{1}{n}=0$.

Series with nonnegative terms

Theorem. A series with nonnegative terms converges if and only if its partial sums form a bounded sequence.

Proof. If $a_n \geq 0$ for all $n \in \mathbb{N}$ then $s_{n+1} = a_{n+1} + s_n \geq s_n$ for all $n \in \mathbb{N}$, so (s_n) is monotonically increasing.

If $\sum_{n=1}^{\infty} a_n$ converges, then (s_n) converges $\Longrightarrow$ (s_n) is bounded.

If (s_n) is bounded, then (s_n) converges since it is monotonically increasing.

Remark. If $a_n \geq 0$ then $\sum_{n=1}^{\infty} a_n$ either converges or its sum is ∞.

Cauchy Condensation Test

Theorem. Suppose $a_1 \geq a_2 \geq a_3 \geq \ldots \geq 0$. Then the series $\sum_{n=1}^{\infty} a_n$ converges if and only if the series $\sum_{k=0}^{\infty} 2^k a_{2^k} = a_1 + 2a_2 + 4a_4 + 8a_8 + \ldots$ converges.

Proof. Let $s_n = a_1 + a_2 + \ldots + a_n = \sum_{k=1}^{n} a_k$ and $t_n = a_1 + 2a_2 + 4a_4 + 8a_8 + \ldots + 2^n a_{2^n} = \sum_{k=1}^{n} 2^k a_{2^k}$

1) (s_n) is monotonically increasing, since the terms of (a_n) are nonnegative and $n \leq 2^n - 1$ for all $n \in \mathbb{N}^+$ so $s_n \leq s_{2^n-1}$. Then

$$s_n \leq s_{2^n-1} = \mathbf{a_1} + (\mathbf{a_2} + \mathbf{a_3}) + (\mathbf{a_4} + \mathbf{a_5} + \mathbf{a_6} + \mathbf{a_7}) + \ldots + (a_{2^{n-1}} + \ldots + a_{2^n-1}) \leq$$
$$\leq \mathbf{a_1} + (\mathbf{a_2} + \mathbf{a_2}) + (\mathbf{a_4} + \mathbf{a_4} + \mathbf{a_4} + \mathbf{a_4}) + \ldots + (a_{2^{n-1}} + \ldots + a_{2^{n-1}}) =$$
$$= \mathbf{a_1} + \mathbf{2\,a_2} + \mathbf{4\,a_4} + \ldots + 2^{n-1} a_{2^{n-1}} =$$
$$= \frac{1}{2}(a_1 + 2a_2 + 4a_4 + 8a_8 + \ldots + 2^n a_{2^n}) = t_{n-1}$$

Assume that $\sum_{k=1}^{n} 2^k a_{2^k}$ is convergent $\Longrightarrow$ (t_n) is convergent, so it is bounded $\Longrightarrow$ (s_n) is bounded above since $s_n \leq s_{2^n-1} \leq t_{n-1}$ $\Longrightarrow$ (s_n) is convergent since it is monotonically increasing.

2) $$s_{2^n} = \mathbf{a_1} + \mathbf{a_2} + (\mathbf{a_3} + \mathbf{a_4}) + (\mathbf{a_5} + \mathbf{a_6} + \mathbf{a_7} + \mathbf{a_8}) + \ldots + (a_{2^{n-1}+1} + \ldots + a_{2^n}) \geq$$
$$\geq \frac{\mathbf{1}}{\mathbf{2}}\mathbf{a_1} + \mathbf{a_2} + (\mathbf{a_4} + \mathbf{a_4}) + (\mathbf{a_8} + \mathbf{a_8} + \mathbf{a_8} + \mathbf{a_8}) + \ldots + (a_{2^n} + \ldots + a_{2^n}) =$$
$$= \frac{\mathbf{1}}{\mathbf{2}}\mathbf{a_1} + \mathbf{a_2} + \mathbf{2\,a_4} + \mathbf{4\,a_8} + \ldots + 2^{n-1} a_{2^n} = \frac{1}{2} t_n$$

Assume that $\sum_{n=1}^{\infty} a_n$ is convergent $\Longrightarrow$ (s_n) is convergent, so it is bounded $\Longrightarrow$ (t_n) is bounded above since $\frac{1}{2} t_n \leq s_{2^n}$ $\Longrightarrow$ (t_n) is convergent since it is monotonically increasing $\Longrightarrow$ $\sum_{k=0}^{\infty} 2^k a_{2^k}$ is convergent.

The p-series (or hyperharmonic series)

Theorem. $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

Proof. 1) If $p \leq 0$ then $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n^p} = \lim_{n\to\infty} n^{|p|} \neq 0$, so by the nth term test, the series diverges.

2) If $p > 0$ then $a_n = \frac{1}{n^p}$ is monotonically decreasing, so the Cauchy condensation theorem is applicable, that is, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ and $\sum_{k=1}^{\infty} 2^k \cdot \frac{1}{(2^k)^p}$ are both convergent or both divergent. Then

$$\sum_{k=1}^{\infty} 2^k \cdot \frac{1}{(2^k)^p} = \sum_{k=1}^{\infty} \frac{1}{2^{-k}} \cdot \frac{1}{2^{k\,p}} = \sum_{k=1}^{\infty} \frac{1}{2^{(p-1)\,k}} = \sum_{k=1}^{\infty} \left(\left(\frac{1}{2}\right)^{p-1}\right)^k.$$

This is a geometric series with ratio $r = \left(\frac{1}{2}\right)^{p-1}$ and it is convergent if and only if

$$|\, r \,| = \left(\frac{1}{2}\right)^{p-1} < 1 \iff p - 1 > 0 \iff p > 1.$$

Examples

1. Is the series $\sum_{n=n_1}^{\infty} \frac{1}{n \cdot \log_2 n}$ convergent or divergent?

Solution. The sequence $a_n = \frac{1}{n \cdot \log_2 n}$ is monotonic decreasing and the terms are nonnegative,

so the Cauchy Condensation Test can be applied.

$\sum_{k=k_1}^{\infty} 2^k \cdot a_{2^k} = \sum_{k=k_1}^{\infty} 2^k \cdot \frac{1}{2^k \cdot \log_2(2^k)} = \sum_{k=k_1}^{\infty} \frac{1}{k}$, this the harmonic series which is divergent

$\Longrightarrow$ the series $\sum_{n=n_1}^{\infty} a_n$ is divergent.

2. Show that $\sum_{n=n_1}^{\infty} \frac{1}{n \cdot (\log_2 n)^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

Solution. If $p > 0$ then the sequence $a_n = \frac{1}{n \cdot (\log_2 n)^p}$ is monotonic decreasing and the terms are nonnegative, so the Cauchy Condensation Test can be applied.

$\sum_{k=k_1}^{\infty} 2^k \cdot a_{2^k} = \sum_{k=k_1}^{\infty} 2^k \cdot \frac{1}{2^k \cdot \log_2(2^k)^p} = \sum_{k=k_1}^{\infty} \frac{1}{k^p}$, this the p-series which converges if $p > 1$ and diverges if $p \leq 1$.

If $p \leq 0$ then for example the comparison test can be used to show divergence (see later).

Then $a_n \geq \frac{1}{n}$ and $\sum_{n=n_1}^{\infty} \frac{1}{n}$ diverges $\Longrightarrow \sum_{n=n_1}^{\infty} a_n$ also diverges.

3. Is the series $\sum_{n=n_1}^{\infty} \frac{1}{n \cdot \log_2 n \cdot \log_2 \log_2 n}$ convergent or divergent?

Solution. The sequence $a_n = \frac{1}{n \cdot \log_2 n \cdot \log_2 \log_2 n}$ is monotonic decreasing and the terms are nonnegative, so the Cauchy Condensation Test can be applied.

$\sum_{k=k_1}^{\infty} 2^k \cdot a_{2^k} = \sum_{k=k_1}^{\infty} 2^k \cdot \frac{1}{2^k \cdot \log_2(2^k) \cdot \log_2(\log_2(2^k))} = \sum_{k=k_1}^{\infty} \frac{1}{k \cdot \log_2 k}$, this is divergent (see example 1.)

$\Longrightarrow$ the series $\sum_{n=n_1}^{\infty} a_n$ is also divergent.