Calculus 1, 1st lecture

Logic. Sets.
https://www.whitman.edu/mathematics/higher_math_online/chapter01.html

Logic

Statements and truth values

A **statement** (or a proposition) is a declarative sentence that has a **truth value** (or logical value), that is, it is either **true** or **false** but not both.

More generally, a **formula** is a statement, possibly containing a few variables, that is either true or false when certain values are assigned to each variable.

Example: If $P(x)$ is $(x+2)(x-3)>0$ then $P(4)$ and $P(5)$ are true, $P(1)$ and $P(2)$ are false.
If $Q(x, y)$ is $x+y=5$ then $Q(2, 3)$ is true, $Q(2, 4)$ is false.

The **universe of discourse** (or universe) is the set that contains everything of interest. For example, it can be the set of real numbers, the set of positive integers, the set of all students of a school etc. The universe of discourse is usually clear from the context but sometimes it has to be clarified explicitly.

Statements can be combined together using **logical connectives** or **logical operations** and can be given by **truth tables**. Most common logical connectives are as follows.

Logical connectives

1) Negation (logical NOT): The statement "not P" or "the denial of P" is true if and only if P is false. Notation: $\neg P$.

2) Conjunction (logical AND): The statement "P and Q" is true if and only if both P and Q are true. Notation: $P \wedge Q$.

3) Disjunction (logical OR): The statement "P or Q" is true if and only if at least one of P and Q are true. Notation: $P \vee Q$.

4) Implication: The statement "if P then Q" or "P implies Q" is true if and only if both P and Q are true or if P is false and Q is arbitrary. Notation: $P \Longrightarrow Q$ (or $P \longrightarrow Q$).

$P:$ hypothesis, premise; $Q:$ conclusion, consequence

5) Biconditional (logical equivalence): The statement "P if and only if Q" is true if and only if both P and Q are true or both are false. Notation: $P \Longleftrightarrow Q$ (or $P \longleftrightarrow Q$). Abbreviation: "P iff Q".

Truth tables (1 = true, 0 = false) :

P	$\neg P$
1	0
0	1

P	Q	$P \wedge Q$	$P \vee Q$	$P \Longrightarrow Q$	$P \Longleftrightarrow Q$
1	1	1	1	1	1
1	0	0	1	0	0
0	1	0	1	1	0
0	0	0	0	1	1

Some identities

The following identities can be proved by truth tables:
1. $\neg(\neg P) = P$
2. commutativity: $P \vee Q = Q \vee P$, $P \wedge Q = Q \wedge P$
3. associativity: $(P \vee Q) \vee R = P \vee (Q \vee R)$, $(P \wedge Q) \wedge R = P \wedge (Q \wedge R)$
4. distributivity: $P \wedge (Q \vee R) = (P \wedge Q) \vee (P \wedge R)$, $P \vee (Q \wedge R) = (P \vee Q) \wedge (P \vee R)$

Necessary and sufficient conditions

1. **Implication**: In the case of the statement $P \Longrightarrow Q$ it is said that "P is a **sufficient condition** for Q" or "Q is a **necessary condition** for P". Other terminologies: "Q only if P", "Q when P", "Q follows from P" etc.
2. **Equivalence**: In the case of the statement $P \Longleftrightarrow Q$ (that is, when both $P \Longrightarrow Q$ and $Q \Longrightarrow P$ hold) it is said that "P is a **necessary and sufficient condition** for Q".

Exercise 1.

Give a

a) necessary but not sufficient

b) sufficient but not necessary

c) necessary and sufficient

condition for the integer N to be divisible by 10.

Solution:

a) N is divisible by 2; or: N is divisible by 5.

b) N is divisible by 20; or: N is divisible by 100;

c) N is divisible by 2 and 5; or: N ends in 0; or: N can be written as $N = 10\,k$ where k is and integer.

Exercise 2.

Give a

a) necessary but not sufficient

b) sufficient but not necessary

c) necessary and sufficient

for a quadrilateral to be a parallelogram. (Homework)

De Morgan's laws

Prove the following identities:

1. $\neg(P \land Q) \equiv \neg P \lor \neg Q$
2. $\neg(P \lor Q) \equiv \neg P \land \neg Q$

Proof of the 1st identity with a truth table (1 = true, 0 = false):

P	Q	$P \land Q$	$\neg(P \land Q)$	$\neg P$	$\neg Q$	$\neg P \lor \neg Q$
1	1	1	0	0	0	0
1	0	0	1	0	1	1
0	1	0	1	1	0	1
0	0	0	1	1	1	1

Example

a) Statement: $x > -2$ and $x < 3$ $(\Longleftrightarrow -2 < x < 3)$
Negation: $x \leq -2$ or $x \geq 3$

b) Statement: I watch TV or read a newspaper.
Negation: I don't watch TV and don't read a newspaper.

The implication and its negation

Prove the following identities:

1. $(P \Longrightarrow Q) \equiv (\neg P \lor Q)$
2. $(P \Longrightarrow Q) \equiv (\neg Q \Longrightarrow \neg P)$
3. $\neg(P \Longrightarrow Q) \equiv P \land \neg Q$

Solution: By truth tables (1 = true, 0 = false):

1. $(P \Longrightarrow Q) \equiv (\neg P \lor Q)$

P	Q	$\neg P$	$P \Longrightarrow Q$	$\neg P \lor Q$
1	1	0	1	1
1	0	0	0	0
0	1	1	1	1
0	0	1	1	1

2. $(P \Longrightarrow Q) \equiv (\neg Q \Longrightarrow \neg P)$

P	Q	$P \Longrightarrow Q$	$\neg Q$	$\neg P$	$\neg Q \Longrightarrow \neg P$
1	1	1	0	0	1
1	0	0	1	0	0
0	1	1	0	1	1
0	0	1	1	1	1

2. Another proof, using part 1.: $(P \Longrightarrow Q) \equiv (\neg P \lor Q) \equiv (Q \lor \neg P) \equiv (\neg(\neg Q) \lor \neg P) \equiv (\neg Q \Longrightarrow \neg P)$

3. The negation of the implication $(P \Longrightarrow Q) \equiv (\neg P \lor Q)$ can be proved by a truth table or by the De Morgan's laws: $\neg(P \Longrightarrow Q) \equiv \neg(\neg P \lor Q) \equiv \neg(\neg P) \land \neg Q \equiv P \land \neg Q$

Example

Statement:	If it is Saturday, then I go to the cinema.
Formally:	$P \Longrightarrow Q$, where P : it is Saturday, Q : I go to the cinema
Reformulation:	$P \Longrightarrow Q \equiv \neg P \lor Q$: It is not Saturday or I go to the cinema.
Negation:	$P \land \neg Q$: It is Saturday and I don't go to the cinema.

Contrapositive

The **contrapositive** of the implication $P \Longrightarrow Q$ is $\neg Q \Longrightarrow \neg P$.

We have seen above that a statement and its contrapositive are logically equivalent.

Example

Statement:	If it is raining, then the ground is wet.
Contrapositive:	If the ground is not wet, then it is not raining.
Statement:	If the clock is ringing, then I get up.
Contrapositive:	If I don't get up, then the clock is not ringing.

The equivalence and its negation

Prove the following identities:
1. $(P \Longleftrightarrow Q) \equiv (P \Longrightarrow Q) \land (Q \Longrightarrow P)$
2. $\neg (P \Longleftrightarrow Q) \equiv (P \land \neg Q) \lor (Q \land \neg P)$

Solution:
1. By truth table (homework).
2. Using the first identity, the De Morgan's laws and the identity $\neg (P \Longrightarrow Q) \equiv P \land \neg Q$ we get:

$$\neg (P \Longleftrightarrow Q) \equiv \neg ((P \Longrightarrow Q) \land (Q \Longrightarrow P)) \equiv \neg (P \Longrightarrow Q) \lor \neg (Q \Longrightarrow P) \equiv (P \land \neg Q) \lor (Q \land \neg P)$$

Remark: From the truth table it can be seen that $P \Longleftrightarrow Q$ is true if and only if P and Q are both true or both false. Thus its negation will be true if and only if one of P and Q is true and the other is false, or vice versa.

Example

Statement:	I go hiking if and only if it is not raining.
Formally:	$P \Longleftrightarrow Q$, where P: I go hiking, Q: it is not raining
Negation:	$(P \land \neg Q) \lor (Q \land \neg P)$, that is:
	I go hiking and it is raining or I don't go hiking and it is not raining.

Tautology and contradiction

A **tautology** is a proposition which is **always true**.
Examples: $P \lor (\neg P)$ or $(P \Longrightarrow Q) \Longleftrightarrow (\neg Q \Longrightarrow \neg P)$.

A **contradiction** s a proposition which is **always false**.
Example: $P \land (\neg P)$.

Quantifiers

The **universal quantifier** is the symbol ∀ and expresses “for all”, “for every”, “given any”.
“$\forall x \in H$, $P(x)$” denotes the statement “for all x in H, $P(x)$”.

Examples

1. The square of any real number is nonnegative: $\forall x \in \mathbb{R}\ (x^2 \geq 0)$, or $\forall x\ (x \in \mathbb{R} \Longrightarrow x^2 \geq 0)$
2. $\forall x \in [0, 1]\ (x^2 \leq x)$, or $\forall x\ (x \in [0, 1] \Longrightarrow x^2 \leq x)$ mean the same.
3. $\forall x \in \mathbb{R}\ (x^2 + 2x + 3 > 0)$
4. $\forall x \forall y\ (x \cdot y = y \cdot x)$
5. All squares are rhombuses: $\forall x$ (x is a square $\Longrightarrow$ x is a rhombus).
6. If a real number is positive then so is its reciprocal: $\forall x \left((x > 0) \Longrightarrow \left(\frac{1}{x} > 0\right)\right)$.

The **existential quantifier** is the symbol ∃ and expresses "there exists", "there is at least one".
"$\exists x \in H$, $P(x)$" denotes the statement "there exists an x in H such that $P(x)$" or "there exists at least one x in H such that $P(x)$" or "for some x, $P(x)$".

Examples

1. $\exists x \in \mathbb{R}\ (x^2 < 1)$, or $\exists x\ (x \in \mathbb{R} \land x^2 < 1)$
2. There exists a rhombus that is not a square: $\exists x$ (x is a rhombus $\land$ x is not a square)
3. There exists a prime number p such that $p + 2$ is also a prime (these are called twin primes):
   $(\exists p \in \mathbb{N})$ (p is prime $\land$ $p + 2$ is prime)
4. $\exists x \exists y\ (x^2 + y^2 = 2)$

Negations of propositions

Statement:	**Negation:**
P or Q	not P and not Q
P and Q	not P or not Q
if P, then Q	P and not Q
For all x, $P(x)$	There exists an x such that not $P(x)$
There exists an x such that $P(x)$	For every x, not $P(x)$

Exercise 3.

Negate the following statements.
a) All windows are open.
b) On each floor there is a window that is open.
c) In every building there is a floor where every window is open.
d) For all positive number ε there exists a positive number δ such that for all real number x, if $|x-a|<\delta$, then $|f(x)-f(a)|<\varepsilon$.
e) Every sailor knows a port where there is a pub he hasn't been to before.

Exercise 4.

Are the following statements true or false? Write down the negation of the statements.

a) $(\forall x\in\mathbb{R})(\exists y\in\mathbb{R})(x+y=0)$

b) $(\exists y\in\mathbb{R})(\forall x\in\mathbb{R})(x+y=0)$

c) $(\forall x\in\mathbb{R})(\exists y\in\mathbb{R})(x<y)$

d) $(\forall x\in\mathbb{R})(\forall y\in\mathbb{R})((x^2=y^2)\Longrightarrow(|x|=|y|))$

e) $(\forall x\in\mathbb{R})(\forall y\in\mathbb{R})(x<y\Longrightarrow x^2<y^2)$

f) $(\forall x\in\mathbb{R})(\forall y\in\mathbb{R})\left(x<y\Longrightarrow \frac{1}{x}>\frac{1}{y}\right)$

g) $(\forall a\in\mathbb{R})(\forall b\in\mathbb{R})(\exists x\in\mathbb{R})(a\cdot x=b)$

h) $(\exists p\in\mathbb{N})((p\text{ is prime})\wedge(p+10\text{ is prime}))$

i) $(\exists x\in\mathbb{Q})(x^2=3)$

j) $[x\in\mathbb{N}\wedge y\in(\mathbb{N}\setminus\{1,x\})]\Longrightarrow \frac{x}{y}\notin\mathbb{N}$

Exercise 5.

Write down the following statement with logical formulas. Is it true or false? Write down the negation:

If a real number is less than every positive number, then it cannot be positive.

Sets

Basic concepts

A set is a collection of objects. Any one of the objects in a set is called a **member** or an **element** of the set. If x is an element of a set A then we write $x\in A$.

Two sets are **equal** if and only if they have the same elements.
Example: $\{1,2\}=\{2,1\}=\{1,1,2\}=\{1,2,2,1,2,1\}$

The **empty set** is the set without elements: $\emptyset=\{\}$.
Note that $\emptyset\neq\{\emptyset\}$: the first contains nothing, the second contains a single element, namely the empty set.

Definition of sets: $\{x\in\text{ universal set }\mid\text{ conditions for }x\}$ or $\{x\in\text{ universal set }:\text{ conditions for }x\}$.

Example: $\{x \in \mathbb{Z} : x > 0\}$: the set of positive integers
$\{x \in \mathbb{Z} : \exists\, n \in \mathbb{Z}\ (x = 2\,n)\}$: the set of even integers

Notations

Real numbers: $\mathbb{R}$
Positive real numbers: $\mathbb{R}^+$
Natural numbers: $\mathbb{N} = \{0, 1, 2, 3, 4, \ldots\}$ (non-negative integers) or
$\mathbb{N} = \{1, 2, 3, 4, \ldots\}$ (positive integers).
Integers: $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$
Positive integers: $\mathbb{N}^+$ (or $\mathbb{Z}^+$)
Rational numbers: $\mathbb{Q} = \left\{x \in \mathbb{R} \mid \exists\, k \in \mathbb{Z}, \exists\, n \in \mathbb{Z} \setminus \{0\}, \left(x = \frac{k}{n}\right)\right\}$

Intervals

$[a; b] = \{x \in \mathbb{R} \mid a \le x \le b\}$ (closed interval)
$[a; b[= [a; b) = \{x \in \mathbb{R} \mid a \le x < b\}$ (interval closed from the left and open from the right)
$]a; b[= (a; b) = \{x \in \mathbb{R} \mid a < x < b\}$ (open interval)

$]a, +\infty[= (a, +\infty) = \{x \in \mathbb{R} \mid a < x\}$
$]-\infty; b] = (-\infty, b] = \{x \in \mathbb{R} \mid x \le b\}$

$\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}^+ = (0, +\infty)$, $\mathbb{R}_0^+ = [0, +\infty)$, $\mathbb{R}^- = (-\infty, 0)$

Subsets

A is a **subset** of B if $\forall x (x \in A \Rightarrow x \in B)$. Notation: $A \subseteq B$.
The sets A and B are **equal** if and only if $A \subseteq B$ and $A \subseteq B$, that is, $\forall x (x \in A \Leftrightarrow x \in B)$.

A is a **proper subset** of B if $A \subseteq B$ and $A \neq B$. (There exists at least one element $a \in A$ such that $a \notin B$.) Notation: $A \subset B$.

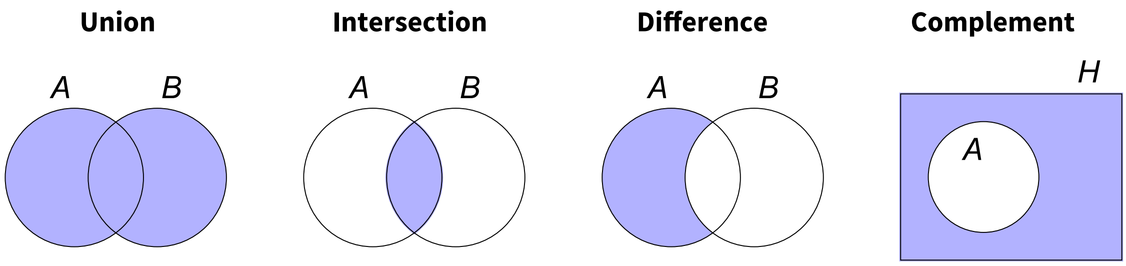
Example: $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$

Operations with sets

The **union** of A and B: $A \cup B = \{x \mid x \in A \lor x \in B\}$
The **intersection** of A and B: $A \cap B = \{x \mid x \in A \land x \in B\}$
The **difference** of A and B: $A \setminus B = \{x \mid x \in A \land x \notin B\}$

The **complement** of A contains all the elements in a universal set H that are not included in A:
$A^c = \overline{A} = \{x \in H \mid x \notin A\} = H \setminus A$ or $\overline{A} = \{x \mid x \notin A\}$.

A and B are **disjoint** if $A \cap B = \emptyset$.

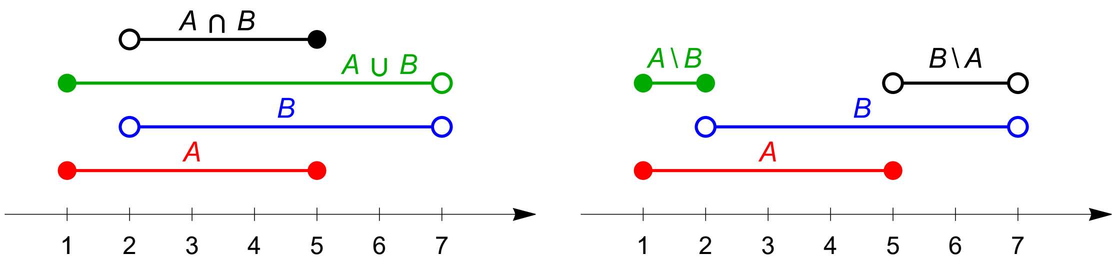


Example

Let $A = [1, 5]$ and $B = (2, 7)$ $\Longrightarrow$ $A \cup B = [1, 7)$ $\quad A \setminus B = [1, 2]$

$A \cap B = (2, 5]$ $\quad B \setminus A = (5, 7)$

If the universal set is $H = \mathbb{R}$ then $\overline{A} = (-\infty, 1) \cup (5, \infty)$ and $\overline{B} = (-\infty, 2] \cup [7, \infty)$.



Cartesian product

In the **ordered pair** (a, b), the first entry is a and the second entry is b. These are also called first and second components or coordinates.

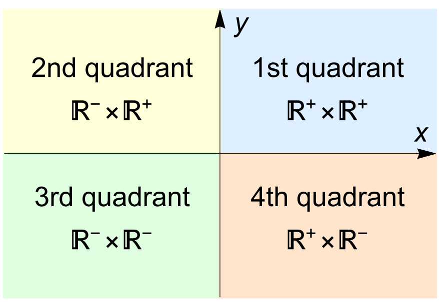
The order of the term matters: $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$.
For example, $(1, 2) \neq (2, 1)$.

Remark: $(a, b) = \{\{a\}, \{a, b\}\}$.

The **Cartesian product** of the sets A and B: $A \times B = \{(a, b) \mid a \in A \wedge b \in B\}$

Examples:

- $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$: the plane
- $\mathbb{R}^+ \times \mathbb{R}^+$: 1st quadrant, $\mathbb{R}^- \times \mathbb{R}^+$: 2nd quadrant etc.
- $\mathbb{R} \times \mathbb{R} \times \mathbb{R} = \mathbb{R}^3$: the 3-dimensional space.



Some identities

1) Commutativity: $A \cap B = B \cap A$, $A \cup B = B \cup A$
2) Associativity: $(A \cap B) \cap C = A \cap (B \cap C)$, $(A \cup B) \cup C = A \cup (B \cup C)$
3) Distributivity: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

4) De Morgan's laws: $\overline{A \cap B} = \overline{A} \cup \overline{B}$, $\overline{A \cup B} = \overline{A} \cap \overline{B}$

5) $A \cap B \subseteq A$ 6) $A \subseteq A \cup B$ 7) $A \subseteq B \iff \overline{B} \subseteq \overline{A}$

Families of sets

https://www.whitman.edu/mathematics/higher_math_online/section01.06.html

Definition: Assume that $I \neq \emptyset$ is a set, called the **index set**, and with each $i \in I$ we associate a set A_i. Then $\{A_i : i \in I\}$ is called an **indexed family of sets**. It is also denoted as $\{A_i\}_{i \in I}$.

Definition: If $\{A_i : i \in I\}$ is an indexed family of sets then
- the intersection of the sets A_i is: $\bigcap_{i \in I} A_i = \{x : \forall\, i \in I\ (x \in A_i)\}$
- the union of the sets A_i is: $\bigcup_{i \in I} A_i = \{x : \exists\, i \in I\ (x \in A_i)\}$

Examples.

1. Suppose that I is the days of the year, and for each $i \in I$, A_i is the set of people whose birthday is i. Then $\bigcap_{i \in I} A_i$ is the empty set and $\bigcup_{i \in I} A_i$ is the set of all people.

2. Suppose I is the set of integers and for each $i \in I$, A_i is the set of multiples of i, that is, $A_i = \{x \in \mathbb{Z} : i \mid x\}$. The notation $i \mid x$ means that x is a multiple of i or i divides x.
$\Longrightarrow A_0 = \{0\}$, $A_1 = A_{-1} = \mathbb{Z}$, $A_2 = A_{-2} = \{\ldots, -4, -2, 0, 2, 4, \ldots\}$, $A_3 = A_{-3} = \{\ldots, -6, -3, 0, 3, 6, \ldots\}$ etc.
Then $\bigcap_{i \in I} A_i = \{0\}$ and $\bigcup_{i \in I} A_i = \mathbb{Z}$.

3. Suppose $I = [0, 1] \subset \mathbb{R}$ and for each $i \in I$, let $A_i = (i - 1, i + 1) \subset \mathbb{R}$.
Then $\bigcap_{i \in I} A_i =]0; 1[$ and $\bigcup_{i \in I} A_i =]0; 2[$.

Theorem (De Morgan's laws): If $\{A_i : i \in I\}$ is an indexed family of sets then

a) $\left(\bigcap_{i \in I} A_i\right)^c = \bigcup_{i \in I} A_i^c$ and b) $\left(\bigcup_{i \in I} A_i\right)^c = \bigcap_{i \in I} A_i^c$

Theorem: If $\{A_i : i \in I\}$ is an indexed family of sets and B is any set then

a) $\bigcap_{i \in I} A_i \subseteq A_j$, for each $j \in I$.

b) $A_j \subseteq \bigcup_{i \in I} A_i$, for each $j \in I$.

c) if $B \subseteq A_i$, for all $i \in I$, then $B \subseteq \bigcap_{i \in I} A_i$.

d) if $A_i \subseteq B$, for all $i \in I$, then $\bigcup_{i \in I} A_i \subseteq B$.

Solutions

Solution, exercise 3.

a) Statement: **All** windows **are open**. (**Each** / **Every** window **is open**.)
Negation: **There is** a window that **is closed**.

b) Statement: On **every** floor **there is** a window that **is open**.
Negation: **There is** a floor where **every** window **is closed**.

c) Statement: In **every** building **there is** a floor where **every** window **is open**.
Negation: **There is** a building where on **each** floor **there is** a window that **is closed**.

d) Statement: **For all** positive number ε **there exists** a positive number δ such that **for all** real number x, **if** $|x-a| < \delta$, **then** $|f(x)-f(a)| < \varepsilon$.

Negation: **There exists** a positive number ε, such that **for all** positive number δ **there exists** a real number x, such that $|x-a| < \delta$ **and** $|f(x)-f(a)| \geq \varepsilon$.

e) Statement: Every sailor knows a port where there is a pub he hasn't been to before.
Reformulation: **For every** sailor **there exists** a port where **there is** a pub he **hasn't been to before**.
Negation: **There is** a sailor who **has already been to every** pub in **every** port.

Solution, exercise 4.

a) Statement: $(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})(x+y=0)$. It is true, since it holds for $y=-x$.
Negation: $(\exists x \in \mathbb{R})(\forall y \in \mathbb{R})(x+y \neq 0)$. It is false.

b) Statement: $(\exists y \in \mathbb{R})(\forall x \in \mathbb{R})(x+y=0)$. It is false, since if there existed such a y, then choosing $x = 1-y$ would give a contradiction: $x+y=(1-y)+y=1 \neq 0$.
Negation: $(\forall y \in \mathbb{R})(\exists x \in \mathbb{R})(x+y=0)$.

Remark: Examples a) and b) show that the order of the quantifiers is essential.

c) Statement: $(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})(x<y)$, that is, there is no largest real number. It is true.
Negation: $(\exists x \in \mathbb{R})(\forall y \in \mathbb{R})(x \geq y)$. It is false.

d) Statement: $(\forall x \in \mathbb{R})(\forall y \in \mathbb{R})\left((x^2=y^2) \Longrightarrow (|x| = |y|)\right)$, that is, if the squares of any two real numbers are equal then they have the same absolute values. It is true.
Negation: $(\exists x \in \mathbb{R})(\exists y \in \mathbb{R})\left((x^2=y^2) \wedge (|x| \neq |y|)\right)$. It is false.

e) Statement: $(\forall x \in \mathbb{R})(\forall y \in \mathbb{R})\left(x<y \Longrightarrow x^2<y^2\right)$. It is false, for example if $x=-2$, $y=1$.
Negation: $(\exists x \in \mathbb{R})(\exists y \in \mathbb{R})\left((x<y) \wedge (x^2 \geq y^2)\right)$. It is true.

Remark: The following statement is true: $(\forall x \in \mathbb{R})\,(\forall y \in \mathbb{R})\,(0 < x < y \implies x^2 < y^2)$

f) Statement: $(\forall x \in \mathbb{R})\,(\forall y \in \mathbb{R})\,\left(x < y \implies \frac{1}{x} > \frac{1}{y}\right)$. It is false, for example if $x = -2,\ y = 3$.

Negation: $(\exists x \in \mathbb{R})\,(\exists y \in \mathbb{R})\,\left((x < y) \wedge \left(\frac{1}{x} \leq \frac{1}{y}\right)\right)$. It is true.

Remark: The following statement is true: $(\forall x \in \mathbb{R})\,(\forall y \in \mathbb{R})\,\left(0 < x < y \implies \frac{1}{x} > \frac{1}{y}\right)$

g) Statement: $(\forall a \in \mathbb{R})\,(\forall b \in \mathbb{R})\,(\exists x \in \mathbb{R})\,(a \cdot x = b)$.

It is false, for example if $a = 0$ and $b = 1$ then $(\forall x \in \mathbb{R})\,(a \cdot x = 0 \neq 1 = b)$

Negation: $(\exists a \in \mathbb{R})\,(\exists b \in \mathbb{R})\,(\forall x \in \mathbb{R})\,(a \cdot x \neq b)$. It is true.

h) Statement: $(\exists p \in \mathbb{N})$ ((p is prime) $\wedge$ ($p + 10$ is prime)).

It is true, for example 3, 13 or 7, 17 are such pairs of primes.

Negation: $(\forall p \in \mathbb{N})$ ((p is not prime) $\vee$ ($p + 10$ is not prime)). It is false.

i) Statement: $(\exists x \in \mathbb{Q})\,(x^2 = 3)$. It is false.

Negation: $(\forall x \in \mathbb{Q})\,(x^2 \neq 3)$. It is true. (See the proof later.)

j) Statement: $[x \in \mathbb{N} \wedge y \in (\mathbb{N} \setminus \{1, x\})] \implies \frac{x}{y} \notin \mathbb{N}$

Reformulation: $(\forall x \in \mathbb{N})\,(\forall y \in \mathbb{N} \setminus \{1, x\})\left(\frac{x}{y} \notin \mathbb{N}\right)$. It is false, for example if $x = 6$, $y = 2$.

Negation: $(\exists x \in \mathbb{N})\,(\exists y \in \mathbb{N} \setminus \{1, x\})\left(\frac{x}{y} \in \mathbb{N}\right)$

Solution, exercise 5.

Statement: If a real number is less than every positive number, then it cannot be positive. It is true.

Formally: $(\forall x)\ \big(((\forall y > 0)\,(x < y)) \implies (x \leq 0)\big)$

Proof. Assume that x is a real number that is less than every positive number.

Suppose to the contrary that $x > 0$.

- Then $x < x$, which is a contradiction.
- Or: Then $0 < \frac{x}{2} < x$, so x is not less than every positive number, which is a contradiction.

$\implies$ The original statement is true, that is, $x \leq 0$.

Remark: The use of brackets is essential here. The meaning of the following statement is different:

$(\forall x)\ (\forall y > 0)\ \big((x < y) \implies (x \leq 0)\big)$

It means that if a real number is less than a positive number then it cannot be positive.

This is false, since $x < y$ holds for $x = 1 > 0$, $y = 2$.