Practice exercises 4.

The limits $\sqrt[n]{p} \overset{n\to\infty}{\longrightarrow} 1 \ (p > 0)$ and $\sqrt[n]{n} \overset{n\to\infty}{\longrightarrow} 1$

1. Calculate the following limits:

a) $a_n = \sqrt[2n]{2n}$

b) $a_n = \sqrt[n]{2n}$

c) $a_n = \sqrt[2n]{n}$

d) $a_n = \sqrt[5n]{3n}$

e) $a_n = \sqrt[n]{2n^3 + 3}$

f) $a_n = \sqrt[n]{\dfrac{2n^2 + 6}{3n^2 + 2n}}$

g) $a_n = \sqrt[n]{\dfrac{5n^2 + 4n - 5}{n^3 + 6n^2 - n}}$

h) $a_n = \sqrt[n^2]{n}$

i) $a_n = \sqrt[n]{n^3 - n^2 + 4n + 1}$

j) $a_n = \sqrt[n]{3^n + 2^n}$

k) $a_n = \sqrt[n]{3^n - 2^n}$

l) $a_n = \sqrt[n]{4^n + n^2 + 3n}$

m) $a_n = \sqrt[2n+1]{n^2 + \cos n}$

n) $a_n = \sqrt[n]{1 + \dfrac{1}{2} + ... + \dfrac{1}{n}}$

Theorems

2. a) Prove that if for all $n \in \mathbb{N}$ $a_n \geq 0$ and $\lim_{n\to\infty} a_n = A$ then $\lim_{n\to\infty} \sqrt[n]{a_n} = 1$ if $A > 0$.

b) Give examples to show that the limit may change if $A = 0$.

3. a) Prove that if $\lim_{n\to\infty} a_n = A$ then $\lim_{n\to\infty} a_n^n = \begin{cases} 0, & \text{if } |A| < 1 \\ \infty, & \text{if } A > 1 \\ \text{does not exist}, & \text{if } A \leq -1 \end{cases}$.

b) Give examples to show that the limit may change or doesn't exist if $A = 1$ or $A = -1$.

Recursive sequences

6. Investigate the convergence of the following sequences and calculate the limit if it exists.

a) $a_1 = 6, \ \ a_{n+1} = 5 - \dfrac{6}{a_n}, \ \ n = 1, 2, ...$

b) $a_1 = \dfrac{4}{3}, \ \ a_{n+1} = \dfrac{3 + a_n^2}{4}, \ \ n = 1, 2, ...$

c) $a_1 = 1, \ \ a_{n+1} = \sqrt{6 + a_n}, \ \ n = 1, 2, ...$

d) $a_1 = -3, \ \ a_{n+1} = \dfrac{5 - 6a_n^2}{13}, \ \ n = 1, 2, ...$

e) $a_1 = 1, \ \ a_{n+1} = \dfrac{a_n}{1 + a_n}, \ \ n = 1, 2, ...$

f) $b_1 = 2, \ \ b_{n+1} = 4 + \sqrt{b_n - 2} - \dfrac{4}{\sqrt{n+4}}$

7.* Let $A > 0, x_1 = 1$ and $x_{n+1} = \dfrac{x_n + \dfrac{A}{x_n}}{2}$. Prove that $x_n \longrightarrow \sqrt{A}$.

The limit $\left(1 + \frac{x}{n}\right)^n \xrightarrow{n \to \infty} e^x$

4. Calculate the limits of the following sequences.

a) $a_n = \left(1 + \dfrac{1}{6\,n^2}\right)^{6\,n^2+2}$, $\quad b_n = \left(\dfrac{n+5}{n-4}\right)^{n+3}$, $\quad c_n = \left(\dfrac{3\,n+5}{3\,n-4}\right)^{3\,n}$, $\quad d_n = \left(\dfrac{3\,n-1}{3\,n+2}\right)^{2\,n}$

b) $a_n = \left(1 + \dfrac{1}{n}\right)^{n^2}$, $\quad b_n = \left(1 + \dfrac{1}{n^2}\right)^{n}$, $\quad c_n = \left(1 - \dfrac{1}{n^4}\right)^{n^3}$, $\quad d_n = \left(1 - \dfrac{1}{n^4}\right)^{n^5}$

c) $a_n = \left(0.9 + \dfrac{1}{n}\right)^{n}$, $\quad b_n = \left(2 - \dfrac{1}{n}\right)^{n}$, $\quad c_n = \left(\dfrac{4\,n+1}{7\,n+5}\right)^{n}$, $\quad d_n = \left(\dfrac{6\,n+1}{4\,n+5}\right)^{n}$

d) $a_n = \left(\dfrac{3\,n^2+1}{3\,n^2-2}\right)^{3\,n^2}$, $\quad b_n = \left(\dfrac{3\,n^2+1}{3\,n^2-2}\right)^{3\,n^3}$, $\quad c_n = \left(\dfrac{3\,n^2-2}{3\,n^2+1}\right)^{3\,n^3}$, $\quad d_n = \left(\dfrac{3\,n^2-1}{3\,n^2+2}\right)^{3\,n}$

5.* Using the formula $\left(1 + \dfrac{1}{z_n}\right)^{z_n} \longrightarrow e$ (for $z_n \longrightarrow \infty$) evaluate the following limits:

a) $a_n = \left(\dfrac{2^n+3}{2^n+1}\right)^{n}$ $\quad$ b) $a_n = \left(\dfrac{n^2-n+1}{n^2+n+1}\right)^{n}$ $\quad$ c) $a_n = \left(\dfrac{n^2+3\,n-4}{n^2-n+2}\right)^{4\,n+1}$ $\quad$ d) $a_n = \left(\dfrac{n^2+\sqrt{n}+1}{n^2-1}\right)^{4\,n^2+3}$

Limit superior and limit inferior

8. Find the limit inferior and limit superior of the following sequences.

a) $a_n = \dfrac{(-1)^n}{n} + \dfrac{1 + (-1)^n}{2}$

b) $a_n = \left(3 + (-1)^n\right) n$

c) $a_n = 1 + 2\,(-1)^n + 3\,(-1)^{\frac{n\,(n+1)}{2}}$

d) $a_n = \cos\left(\dfrac{n\,\pi}{2}\right) \cdot \dfrac{2\,n^2-3}{n^2+n+8}$, $\quad b_n = \cos\left(\dfrac{n\,\pi}{2}\right) \cdot \dfrac{2\,n^2-3}{n^3+n+8}$

e) $a_n = \dfrac{(-3)^n+8}{5+4^n}$, $\quad b_n = \dfrac{(-4)^n+8}{5+4^n}$

f) $a_n = \sqrt{\dfrac{n^3+(-1)^n\,n^3}{3\,n^3+n+7}}$

Additional exercises

9)* For all $r > 0$ show examples for sequences $a_n \longrightarrow 0+$ and $b_n \longrightarrow 0$ such that

$$a_n{}^{b_n} \longrightarrow r.$$

10)* For all $n \in \mathbb{N}$ we define the value of a_n by placing a decimal point in front of the index n written in the decimal number system and then a zero digit in front of it, and we interpret the number thus obtained in the decimal number system. For example $a_{4523} = 0.4523$ and $a_{100} = 0.100$. Find the accumulation points of the number sequence (a_n).

11)* Consider the following number
sequence:

$$\frac{0}{1}, \quad \frac{0}{2}, \frac{1}{2}, \quad \frac{0}{3}, \frac{1}{3}, \frac{2}{3}, \quad \frac{0}{4}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4} \quad \cdots \quad \frac{0}{n}, \frac{1}{n}, \frac{2}{n}, \cdots, \frac{n-1}{n}, \quad \frac{0}{n+1}, \frac{1}{n+1}, \cdots$$

a) Is 0 an accumulation point of the sequence?
b) Is 1 an accumulation point of the sequence?
c) Is the sequence convergent?
d) Exactly what real numbers are the accumulation points of the sequence?
Give reasons for your answers.

12)* Is there a number sequence whose real accumulation points are
a) the integers;
b) the rational numbers;
c) the points of the set $\left\{1, \frac{1}{2}, \frac{1}{3}, \ldots\right\}$;
d) the points of the set $\left\{1, \frac{1}{2}, \frac{1}{3}, \ldots\right\} \cup \{0\}$;
e) the [0, 1] closed interval?
If the answer is yes, then construct such a sequence.