Practice exercises 3.

1. Find the limit of (a_n) and using the definition, provide a threshold index N for a given $\varepsilon > 0$.

   a) $a_n = \dfrac{7n+4}{2n-1}$ b) $a_n = \dfrac{3n^2+4n+7}{n^2+n+1}$

   c) $a_n = \dfrac{n^2-10^8}{5n^6+2n^3-1}$ d) $a_n = 1+(-1)^{n+1}\cdot 2^{3-n}$

2. Using the definition of the limit, show that

   a) $\lim\limits_{n\to\infty}\left(6n^3+3n\right) = \infty$ b) $\lim\limits_{n\to\infty}\sqrt{n^2-n} = \infty$ c) $\lim\limits_{n\to\infty}\left(n^3-3n^2+5n+9\right) = \infty$

   d) $\lim\limits_{n\to\infty}\dfrac{n^3+3n}{n^2+2} = \infty$ e) $\lim\limits_{n\to\infty}\dfrac{1+n^2-3n^3}{n^2+3n+7} = -\infty$

3. Formulate the following statements without negation:

   a) $\lim\limits_{n\to\infty} a_n \neq A \in \mathbb{R}$

   b) (a_n) is divergent

4. Calculate the limit of the following sequences:

   a) $a_n = \dfrac{n+3}{4n^2+7n+6}$ b) $a_n = \dfrac{n-5n^4}{n^4+8n^3+1}$ c) $a_n = \dfrac{1-n^3}{70-n^2+n}$

   d) $a_n = \dfrac{-n^7+n^6-3}{n^5-n^2+2}$ e) $a_n = \dfrac{(2n^3+3)^2}{(3n+6)^6}$ f) $a_n = \dfrac{(n+1)!}{(3-2n)\,n!}$

   g) $a_n = \dfrac{\binom{n}{2}}{\binom{n}{3}}$ h) $a_n = \dfrac{\binom{2n}{4}}{\binom{n+1}{2}\binom{n-1}{2}}$ i) $a_n = \sqrt[3]{\dfrac{2n^2+6}{3n^2+2n}}$

   j) $a_n = \dfrac{n^{3/2}+n^2+1}{\sqrt{1+n^2}+2\sqrt{n^3+2}}$ k) $a_n = \dfrac{\sqrt[4]{n^3+6}}{\sqrt[3]{n^2+3n+2}}$ l) $a_n = \dfrac{9\sqrt[3]{n}-3\sqrt{2n}+1}{\sqrt[4]{n}+\sqrt{3n}}$

5. Decide whether the following sequences converge and if so, find their limit:

   a) $a_n = \sqrt{n^2+n+1}-\sqrt{n^2-n+1}$ b) $a_n = \sqrt{n^2-7n+1}-\sqrt{n^2-n+4}$

   c) $a_n = \sqrt{2n^2+3n+1}-\sqrt{n^2+1}$ d) $a_n = (3n+1)\left(n-\sqrt{n^2+1}\right)$

   e) $a_n = \dfrac{1}{n-\sqrt{n^2+3n+5}}$ f) $a_n = \sqrt[3]{n^3+3n^2-1}-\sqrt[3]{n^3-2n^2+3n+2}$

6. Prove that $\lim\limits_{n\to\infty}\dfrac{a^n}{n!} = 0$, where $a \in \mathbb{R}$.

7. Calculate the limit of the following sequences:

a) $a_n = \dfrac{\sin(n)}{n}$ b) $a_n = \dfrac{n^2 - 5}{2n^3 + 6n} \cos(n^4 + 5n + 8)$

c) $a_n = \dfrac{\log(n+1)}{n}$ d) $a_n = \dfrac{\log_{10}(n^2) + 3}{\log_3(n)}$ e) $a_n = \dfrac{(-3)^{n+1} + 2^{2n+3}}{8 + 5^n}$

f) $a_n = \dfrac{3^{2n} + n^2 + 1}{3^n + 9^n}$ g) $a_n = \dfrac{7^n + n^7 + 7}{4^n + 3n^2 + 5}$ h) $a_n = \dfrac{4^{n-1} + n^5 \cdot 3^{n+2}}{2^{2n+3} + 2^{n-2}}$

i) $a_n = \dfrac{n^3 2^n + 3^n}{2^{2n} - 3n^2}$ j) $a_n = \dfrac{2n! + n^{20}}{n^n}$ k) $a_n = \dfrac{(2^n + 7^n)^2}{n!}$

8. True or false?

a) If $a_n \longrightarrow A$ then $a_n^2 \longrightarrow A^2$. b) If $a_n^2 \longrightarrow A^2$ then $a_n \longrightarrow A$.

c) If $a_n > 0$ and $b_n \longrightarrow \infty$ then $a_n b_n \longrightarrow \infty$. d) If $a_n \longrightarrow 0$ then $\dfrac{1}{a_n} \longrightarrow \infty$.

e) If $a_n \longrightarrow \infty$ then $\dfrac{1}{a_n} \longrightarrow 0$. f) If $a_n > 0$ and (a_n) is convergent then $\lim\limits_{n \to \infty} a_n > 0$.

9. Is there a convergent sequence that can be written in the following form?

a) $a_n + b_n$ where both (a_n) and (b_n) are divergent

b) $a_n + b_n$ where (a_n) is convergent and (b_n) is divergent

c) $a_n \cdot b_n$ where both (a_n) and (b_n) are divergent

d) $a_n \cdot b_n$ where (a_n) is convergent and (b_n) is divergent

10. Prove the following statements:

a) If $a_n \longrightarrow A$ then $\sqrt[3]{a_n} \longrightarrow \sqrt[3]{A}$.

b) If $c_n \longrightarrow C > 0$ and $d_n \longrightarrow \infty$ then $c_n d_n \longrightarrow \infty$.