Practice exercises 2.

Supremum, infimum

1. Are the following sets bounded below or above? If so, determine their supremum and infimum. Do these sets have a minimum or a maximum?

   a) $H=\left\{\frac{1}{n}+(-1)^n:\ n\in\mathbb{N}\right\}\subset\mathbb{R}$

   b) $H=\left\{\frac{(-1)^n}{n}+1: n\in\mathbb{N}\right\}\subset\mathbb{R}$

   c) $H=\left\{\frac{1}{2^n}+\frac{1}{2^m}: m,\ n\in\mathbb{N}\right\}\subset\mathbb{R}$

   d) $H=\left\{\frac{x^2+1}{3x^2+2}: x\in\mathbb{R}\right\}\subset\mathbb{R}$

   e) $H=\left\{\frac{2x+3}{3x+1}: x\in\mathbb{Z}\right\}\subset\mathbb{R}$

   f) $H=\left\{\frac{x}{y}: 0<x<1,\ 0<y<1\right\}\subset\mathbb{R}$

   g) $H=\left\{\frac{p}{q}: p,\ q\in\mathbb{Z},\ p^2<q^2\right\}\subset\mathbb{R}$

   h) $H=\left\{r\in Q^+ : r^2<2\right\}\subset\mathbb{R}$

2.* Let $\omega\in\mathbb{R}$ be a positive irrational number. Let

   $$A=\{m+n\,\omega:\ m+n\,\omega>0,\ \ m,\ n\in\mathbb{Z}\}$$

   Prove that $\inf A=0$.

3. Let $A\subset\mathbb{R}$ be a nonempty set that is bounded above. Show that the set $B=\{-a : a\in A\}$ is bounded below and $\inf B=-\sup A$.

4. Let $A,\ B\subset\mathbb{R}$ be nonempty bounded sets. Prove that

   a) $\inf(A\cup B)=\min\{\inf A,\ \inf B\}$

   b) $\sup(A\cup B)=\max\{\sup A,\ \sup B\}$

   c) if $A\cap B\neq\emptyset$ then $\inf(A\cap B)\geq\max\{\inf A,\ \inf B\}$

   d) if $A\cap B\neq\emptyset$ then $\sup(A\cap B)\leq\min\{\sup A,\ \sup B\}$

   e) if $A\subset B$ then $\inf A\geq\inf B$ and $\sup A\leq\sup B$

Complex numbers

1. Find the algebraic form ($a+i\,b$) of the following complex numbers:

   a) $\frac{3-2i}{-2+i}$ b) $\frac{2+i}{i(1-4i)}$ c) $i^2,\ i^3,\ i^4,\ \ldots,\ i^{2021}$ d) $(2+i)^{37}\,(2-i)^{38}$

2. Find the trigonometric form ($r\cos\varphi+i\,r\sin\varphi$) of the following complex numbers:

   a) $\sqrt{6}-i\sqrt{2}$ b) $-4i$ c) 8

3. Find the algebraic (or trigonometric) form of the following complex numbers:

   a) $(1+i)^8$ b) $(1-i)^4$ c) $\left(1+i\sqrt{3}\right)^{100}$ d) $\sqrt{i}$ e) $\sqrt[3]{1}$

f) $\sqrt[4]{-16}$ g) $\frac{\sqrt{i}}{1-i}$ h) $\sqrt{-5+12\,i}$

4. Plot the following sets on the complex plane:

a) $\{z \in \mathbb{C}: \ |z-3| \leq 2\}$

b) $\{z \in \mathbb{C}: \ |z|^2 = \text{Re}(z)\}$

c) $\{z \in \mathbb{C}: \ \text{Re}(z+1) = |z-1|\}$

d) $\{z \in \mathbb{C}: \ |z| > 5, \ \text{Im}(z) \geq \text{Re}(z)\}$

e) $\{z \in \mathbb{C}: \ |z| < |z-2\,i|, \ |z-i| \leq 1\}$

f) $\{z \in \mathbb{C}: \ |z+i| + |z+3| = 7\}$

*g)** $\left\{z \in \mathbb{C}: \ \left|\frac{z-1}{z+i}\right| = 2\right\}$

5. Solve the following equations and equation systems on the set of complex numbers:

a) $z^6 + 16\,z^2 = 0$

b) $(3-i)\,z^2 + 3\,i\,z + 6 - i = 0$

c) $|z| - z = 1 + 2\,i$

d) $z^2 = \overline{z}$

e) $z^2 + (1+i)\,\overline{z} + 4\,i = 0$

f) $2\,i\,z^3 = (1+i)^8$

g) $\text{Re}(z) + 2\,\text{Im}(z) = 0$ and $\text{Re}(z^2) - 2\,\text{Im}(z) = 1$

h) $\text{Re}(z^2) = 2\,\text{Im}(z)$ and $\text{Im}(z^2) = 2\,\text{Re}(z)$

6. Assume that for the complex number z, $\text{Im}(z) \neq 0$ and $\text{Im}\left(z + \frac{1}{z}\right) = 0$. Find $|z|$.

7. For which values of n $(n \in \mathbb{N})$ is the complex number $\left(\sqrt{3} - i\right)^n$ real?

8. How many complex roots can a seventh-order real-coefficient polynomial have?

9. One vertex of a regular hexagon is $2 + i$, its center is $3 + 2\,i$. Find the other vertices.

10.* Prove that the sum of the vectors pointing from the center of a regular n-sided polygon to its vertices is the zero vector.

11.* Without solving the following equations, decide how many solutions they have on the set of complex numbers. Give a reason for your answer.

a) $\frac{1+i}{(z+i)^3} = (\overline{z} - i)^3$ b) $(\overline{z} + 1)^7 = \frac{1}{(\overline{z})^7} - 1$

12.* It is an interesting fact that a product of two sums of squares is itself a sum of squares. For example,

$$\left(1^2 + 2^2\right)\cdot\left(3^2 + 4^2\right) = 125 = 5^2 + 10^2 = 2^2 + 11^2$$

Show that for any two pairs of integers $\{a, \ b\}$ and $\{c, \ d\}$, we can find integers $u, \ v$ with

$$\left(a^2 + b^2\right)\left(c^2 + d^2\right) = u^2 + v^2$$