Practice exercises 10.

Differentiation

1. a) Let $f(x) = \sqrt[3]{x}$. Use the definition of the derivative to show that $f'(x) = \frac{1}{3} x^{-\frac{2}{3}}$ if $x \neq 0$.

   Prove that $f'(0)$ does not exist.

   b) Prove that $(\cos x)' = -\sin x$.

2. Differentiate the following functions:

a) $f(x) = \frac{1}{x^3}$ b) $f(x) = (x^2+1)^{17}$ c) $f(x) = \sqrt{\frac{1}{x} \cdot \sqrt[3]{x}}$

d) $f(x) = \sqrt[4]{3x^2+5x}$ e) $f(x) = \tan x$ f) $f(x) = (x^3+3x)(\sin x + \cos x)$

g) $f(x) = \cos(x^3+3x-1)$ h) $f(x) = \cos(2x)\sin(x^2-1)$ i) $f(x) = \sin^5(x^3)$

j) $f(x) = \tan(x^2+1)\sin\frac{1}{x}$ k) $f(x) = \cot\left(\frac{x^2+3}{\sqrt{\sin(2x-1)+7}}\right)$ l) $f(x) = \tan 3x \cos 5x \sin 7x$

3. Differentiate the following functions.

a) $\ln\sqrt{\cos x}$ b) $f(x) = \ln\frac{1+\cos x}{1-\sin x}$ c) $f(x) = \ln\sqrt[4]{\sin^3 x \cos^3 x}$

d) $f(x) = e^{3x^4+x+1}\ln(x^2+1)$ e) $f(x) = \frac{\sin(x)\ln(1+\cos^2(x^3))}{x}$

f) $f(x) = \frac{\arctan(3x^2+4)\cos\left(\sqrt{2x+3}\right)}{\log(\sin 3x)}$ g) $f(x) = \arcsin(1-e^{3x}) + \arctan(2^x+1)$

h) $f(x) = \arccos(x^3-x+1)\,e^{\sin\left(\sqrt{x^2+3}\right)}$ i) $f(x) = x^{\sqrt{x}}$ j) $f(x) = x^{x\log x}$

4. Let $f(x) = \sqrt[3]{x^2}\cdot\sin\sqrt[3]{x^2}$. Calculate $f'(x)$. (At $x = 0$ use the definition.)

5. Let $f(x) = \arctan\frac{1+x}{1-x}$ if $x \neq 1$ and $f(1) = \beta$.

a) Is it possible to choose the value of β such that f is continuous at $x = 1$?

b) Calculate $f'(x)$ if $x \neq 1$.

c) $\lim_{x\to 1} f'(x) = ?$ Does $f'(1)$ exist?

6. Prove that the function $f(x) = \begin{cases} x^2 \sin\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ is differentiable on ℝ but f' is not continuous.

7. Choose the values of the parameters such that the following functions be differentiable on ℝ:

a) $f(x) = \begin{cases} x^2 & \text{if } x \leq x_0 \\ ax+b & \text{if } x > x_0 \end{cases}$ b) $f(x) = \begin{cases} e^{2x} & \text{if } x \geq 0 \\ ax^2+bx+c & \text{if } x < 0 \end{cases}$

8. Find the equation of the tangent line to the following functions at x_0:

a) $f(x) = \sin \sqrt{x}$, $x_0 = \pi^2$ b) $f(x) = x^3 - 8x$, $x_0 = 3$ c) $f(x) = e^{\sin x}$, $x_0 = \pi$

9. Does the function $f(x) = x^2 - 1$ has a tangent line that passes through the point (2, 2)? If so, then find the equation of the tangent line.

10. Find the equation of the tangent line to the following curves at P:

a) $x^3 + y^3 - 6xy = 0$, $P(3, 3)$ b) $2xy + \pi \sin y = 2\pi$, $P\left(1, \frac{\pi}{2}\right)$ c) $x \sin 2y = y \cos 2x$, $P\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$

Mean value theorems

11. Prove that the polynomial $f(x) = x^7 + 14x + 3$ has exactly one root.

12.* Prove that the function $f(x) = x^n + ax + b$ has at most two roots if n is even and at most 3 roots if n is odd.

13. Prove the following inequalities:

a) $|\sin x - \sin y| \le |x - y|$, if $x < y$

b) $\dfrac{b-a}{\cos^2 a} < \tan b - \tan a < \dfrac{b-a}{\cos^2 b}$, if $0 < a < b < \dfrac{\pi}{2}$

c) $\sin x \le \dfrac{\sqrt{3}}{2}\left(x - \dfrac{\pi}{6}\right) + \dfrac{1}{2}$, if $\dfrac{\pi}{6} \le x \le \dfrac{\pi}{2}$

d) $\tan x - 1 > 2x - \dfrac{\pi}{2}$, if $0 < x < \dfrac{\pi}{4}$

e) $\log(1 + x) \le x$, if $x \ge 0$

f) $e^x \ge 1 + x$, if $x \ge 0$

L'Hospital's rule

14. Use L'Hospital's rule to evaluate the following limits:

a) $\lim\limits_{x\to 0} \dfrac{\ln(1+x)}{x}$ b) $\lim\limits_{x\to 0} \dfrac{x - \sin x}{x^3}$ c) $\lim\limits_{x\to 0} \dfrac{3^{\sin x} - 1}{x}$ d) $\lim\limits_{x\to 0} \dfrac{x(1-\cos x)}{x - \sin x}$

e) $\lim\limits_{x\to\infty} \dfrac{x e^{\frac{x}{2}}}{e^x + 1}$ f) $\lim\limits_{x\to\infty} \dfrac{\ln x^2}{\sqrt{x}}$ g) $\lim\limits_{x\to 0} \dfrac{\ln x}{1 + \ln \sin x}$ h) $\lim\limits_{x\to 0} (\arcsin x)(\cot x)$

i) $\lim\limits_{x\to -\infty} x^2 e^x$ j) $\lim\limits_{x\to \frac{\pi}{2}} \left(\dfrac{\pi}{2} - x\right) \tan x$ k) $\lim\limits_{x\to 1} \left(\dfrac{x}{x-1} - \dfrac{1}{\ln x}\right)$ l) $\lim\limits_{x\to 0} \left(\dfrac{1}{x^2} - \dfrac{1}{x \sin x}\right)$

m) $\lim\limits_{x\to 0+} x^x$ n) $\lim\limits_{x\to 1+} x^{1/(x-1)}$ o) $\lim\limits_{x\to 0+} (e^x + x)^{\frac{1}{x}}$ p) $\lim\limits_{x\to 0+} (\sin x)^x$

q) $\lim\limits_{x\to -\infty} \dfrac{e^{8x} - 2e^{-3x}}{e^{5x} + e^{-3x}}$ r) $\lim\limits_{x\to\infty} \dfrac{\operatorname{sh}(3x-2)}{\operatorname{ch}(3x+4)}$