Calculus 1, Final exam, Part 1, 2018.12.18. 9.00-10.30

I. Definitions and theorems (each question is worth 3 points)

1. What is the definition of a sequence a_n converging to $A \in \mathbb{R}$?

2. What is the definition of the supremum of an upper-bounded set of real numbers?

3. State the nth term test for divergence of series.

4. State the ratio test for convergence or divergence of series.

5. What does it mean that a series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent?

6. State the Leibniz-rule for alternating series.

7. What is the definition of a function $f(x)$ being continuous at a point $x_0 \in \mathbb{R}$?

8. State Rolle's theorem.

9. What is the definition of a function $f(x)$ being differentiable at a point $x_0 \in \mathbb{R}$?

10. What is the formula for the arc-length of a function $y = f(x)$, $x \in [a, b]$?

11. When do we say that a function $f(x)$ has a local maximum at a point $x_0 \in \mathbb{R}$?

12. State the Newton-Leibniz formula for continuous functions.

II. Proof of a theorem (this question is worth 20 points)

1. State and prove the integral test for the convergence of series.

III. True or false? Indicate at each statement whether it is true or false. Each question is worth 2 points.

1. If $\lim_{n\to\infty} a_n = A \in \mathbb{R}$ then $a_n > A$ can happen only finitely many times.
2. If $f'(x_0) = 0$ then f necessarily has a local maximum or local minimum or a point of inflection at x_0.
3. The Taylor polynomial of order of the function $\sin x$ around 0 is $T(x) = x - \frac{x^2}{2} + \frac{x^3}{3}$.
4. If a function is differentiable at $x_0 \in \mathbb{R}$ then it is necessarily also continuous at x_0.
5. The partial fraction decomposition of $f(x) = \frac{1+x}{(x-1)^3(x+2)^2}$ cannot contain a term $\frac{A}{(x+2)^3}$.
6. If a function f is differentiable on $[0,1]$ and $f'(x) \leq 1$ for all $x \in [0,1]$ then it implies that $f(1) \leq f(0) + 1$.
7. If $f(x)$ is a nonnegative continuous function on $\mathbb{R}$, and $f(x) \geq \frac{1}{x}$ for all $x \geq 1000$, then $\int_1^\infty f(x)$ necessarily diverges.
8. If a function is differentiable everywhere on $\mathbb{R}$, and $|f(9) - f(5)| \leq 2$, then $|f'(x)| \leq \frac{1}{2}$ for some $x \in [5,9]$.
9. If $a_n \geq 0$ and $\lim_{n\to\infty} a_n = 0$ then $\sum_{n=1}^\infty a_n$ necessarily converges.
10. If a nonnegative function $f(x)$ is continuous on $[0,+\infty[$ and $\lim_{x\to+\infty} f(x) \neq 0$ then $\int_1^\infty f(x)dx$ necessarily diverges.
11. If a function is differentiable everywhere on $\mathbb{R}$, and there exists an $x_0 \in [3,4]$ such that $|f'(x_0)| \leq 1$, then $|f(3) - f(4)| \leq 1$.
12. If a_n is a nonnegative sequence and $\lim_{n\to\infty} a_n = 1$ then $\lim_{n\to\infty} a_n^n = 1$ necessarily holds.
13. If a nonnegative sequence a_n is monotonically increasing then $\lim_{n\to\infty} \frac{1}{a_n} = 0$ necessarily holds.
14. If a nonnegative series $\sum_{n=1}^\infty a_n$ converges then $\sum_{n=1}^\infty a_n^2$ also converges.
15. If a nonnegative series $\sum_{n=1}^\infty a_n$ converges then there exists an N such that $\sqrt[n]{a_n} < 1$ for all $n \geq N$.

IV. Examples

Give examples of sequences, series or functions with prescribed properties. (4+5+5 points)

1. Give an example of a sequence a_n such that a_n is convergent, but a_n is not monotonically increasing or decreasing. (4 points)

2. Give an example of a nonnegative series $\sum_{n=1}^{\infty} a_n$ such that $\frac{a_{n+1}}{a_n} < 1$ for all n but $\sum_{n=1}^{\infty} a_n$ does not converge. (5 points)

3. Give an example of a continuous nonnegative function $f(x)$ such that $\int_1^{\infty} f(x)dx$ converges but $\sum_{n=1}^{\infty} f(n)$ diverges. (5 points)