Calculus 1

Number sequences, part 2.

Binomial theorem

Binomial coefficients: $\binom{n}{k} = \dfrac{n!}{k!\,(n-k)!}$ where $k! = 1 \cdot 2 \cdot \ldots \cdot k$ and $0! = 1$.

Meaning: the number of subsets with k elements of a set with n elements.

Binomial theorem: $(a+b)^n = (a+b)\,(a+b) \ldots (a+b) = \displaystyle\sum_{k=0}^{n} \binom{n}{k} a^k\, b^{n-k}$.

Orders of magnitudes

Definition: Suppose that $a_n \xrightarrow{n\to\infty} \infty$ and $b_n \xrightarrow{n\to\infty} \infty$. Then the order of magnitude of (a_n) is smaller than the order of magnitude of (b_n) if $\dfrac{a_n}{b_n} \xrightarrow{n\to\infty} 0$.

Notation: $a_n << b_n$.

Theorem: $n^n >> n! >> a^n >> n^k >> n^{\frac{1}{k}} >> \log n$, where $a > 1$ and $k \in \mathbb{N}^+$. That is,

a) $\displaystyle\lim_{n\to\infty} \frac{n^n}{n!} = \infty$ **b)** $\displaystyle\lim_{n\to\infty} \frac{n!}{a^n} = \infty$, where $a > 1$ **c)** $\displaystyle\lim_{n\to\infty} \frac{a^n}{n} = \infty$, where $a > 1$

d) $\displaystyle\lim_{n\to\infty} \frac{a^n}{n^k} = \infty$, where $a > 1$ and $k \in \mathbb{N}^+$ **e)** $\displaystyle\lim_{n\to\infty} \frac{n}{\log_2 n} = \infty$

Some proofs. a) $\dfrac{n^n}{n!} = \dfrac{n}{1} \cdot \dfrac{n}{2} \cdot \dfrac{n}{3} \cdot \ldots \cdot \dfrac{n}{n-2} \cdot \dfrac{n}{n-1} \cdot \dfrac{n}{n} \ge n \cdot 1 \cdot 1 \cdot \ldots \cdot 1 \cdot 1 \cdot 1 = n \longrightarrow \infty \implies \dfrac{n^n}{n!} \longrightarrow \infty$

b) For example, if $a = 2$, then $\dfrac{n!}{2^n} = \dfrac{n}{2} \cdot \dfrac{n-1}{2} \cdot \dfrac{n-2}{2} \cdot \ldots \cdot \dfrac{3}{2} \cdot \dfrac{2}{2} \cdot \dfrac{1}{2} \ge \dfrac{n}{2} \cdot 1 \cdot 1 \cdot \ldots \cdot 1 \cdot 1 \cdot \dfrac{1}{2} = \dfrac{n}{4} \longrightarrow \infty \implies \dfrac{n!}{2^n} \longrightarrow \infty$

In general, if $a > 1$, then $\dfrac{n!}{a^n} = \dfrac{n}{a} \cdot \dfrac{n-1}{a} \cdot \ldots \cdot \dfrac{[a]+1}{a} \cdot \dfrac{[a]}{a} \cdot \ldots \cdot \dfrac{1}{a} \ge \dfrac{n}{a} \cdot 1 \cdot \ldots \cdot 1 \cdot c = \dfrac{c}{a} \cdot n \longrightarrow \infty \implies \dfrac{n!}{a^n} \longrightarrow \infty$,

where $c = \dfrac{[a]}{a} \cdot \ldots \cdot \dfrac{1}{a}$.

c) We will prove that $\displaystyle\lim_{n\to\infty} \frac{a^n}{n} = \infty$, where $a = 1 + \delta$ and $\delta > 0$. By the binomial theorem,

$(1+\delta)^n = \displaystyle\sum_{k=0}^{n} \binom{n}{k} \delta^k = \binom{n}{0}\delta^0 + \binom{n}{1}\delta^1 + \binom{n}{2}\delta^2 + \ldots + \binom{n}{n}\delta^n \ge \binom{n}{2}\delta^2$, so

$\dfrac{(1+\delta)^n}{n} \ge \dfrac{\binom{n}{2}\delta^2}{n} = \dfrac{n(n-1)}{2n}\,\delta^2 = \dfrac{n-1}{2}\,\delta^2 \longrightarrow \infty \implies \dfrac{a^n}{n} \longrightarrow \infty$, where $a > 1$.

d) We will prove that $\lim\limits_{n\to\infty} \frac{a^n}{n^k} = \infty$, where $a > 1$ and $k \in \mathbb{N}^+$. This is a consequence of case c),

since if $a > 1$ then $\sqrt[k]{a} > 1$ and $\frac{a^n}{n^k} = \left(\frac{\left(\sqrt[k]{a}\right)^n}{n}\right)^k$.

e) Let $a_n = \frac{n}{\log_2 n}$. It can be shown that (a_n) is monotonic increasing (we can prove this later)

and $a_{2^k} = \frac{2^k}{\log_2 2^k} = \frac{2^k}{k} \longrightarrow \infty$. From these two properties it follows that $a_n \longrightarrow \infty$.

Example: $\frac{n^2 - 3^n}{n! + n^4} = \frac{3^n}{n!} \cdot \frac{\frac{n^2}{3^2} - 1}{1 + \frac{n^4}{n!}} \xrightarrow{n\to\infty} 0 \cdot \frac{0-1}{1+0} = 0.$

Theorem. $\lim\limits_{n\to\infty} n^k a^n = 0$, if $|a| < 1$ and $k \in \mathbb{N}^+$.

1st proof. It is a consequence of the following statements:

a) If $a_n \xrightarrow{n\to\infty} \infty$ then $\frac{1}{a_n} \xrightarrow{n\to\infty} 0$.

b) If $a > 1$ and $k \in \mathbb{N}^+$ then $\frac{a^n}{n^k} \xrightarrow{n\to\infty} \infty$.

c) If $\left| a_n \right| \xrightarrow{n\to\infty} 0$ then $a_n \xrightarrow{n\to\infty} 0$.

2nd proof. It is a consequence of the following statements:

(i) $\lim\limits_{n\to\infty} \sqrt[n]{n} = 1$ (see the proof later)

(ii) If $0 < \lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$ then $a_n \xrightarrow{n\to\infty} 0$.

Proof of (ii): If $L \le q < 1$ then there exists $N \in \mathbb{N}$ such that for all $n > N$, $\sqrt[n]{|a_n|} < q$.

Then $0 < |a_n| < q^n \longrightarrow 0$ so by the Sandwich Theorem $a_n \xrightarrow{n\to\infty} 0$.

Using this, if $|a| < 1$ then $\sqrt[n]{|n^k a^n|} = \left(\sqrt[n]{n}\right)^k \cdot \left| a \right| \longrightarrow 1^k \cdot \left| a \right| < 1 \Longrightarrow n^k a^n \longrightarrow 0$.

Example: $\lim\limits_{n\to\infty} \frac{n^2 + 9^{n+1}}{2n^5 + 3^{2n-1}} = ?$

Solution: $\frac{n^2 + 9^{n+1}}{2n^5 + 3^{2n-1}} = \frac{9^n}{9^n} \cdot \frac{n^2\left(\frac{1}{9}\right)^n + 9}{2n^5\left(\frac{1}{9}\right)^n + \frac{1}{3}} \xrightarrow{n\to\infty} \frac{0+9}{0+\frac{1}{3}} = 27.$

We used that $\lim\limits_{n\to\infty} n^k a^n = 0$, if $|a| < 1$. Here $a = \frac{1}{9}$.

Subsequences

Definition. Suppose that $(n_k): \mathbb{N} \longrightarrow \mathbb{N}$ is a strictly monotonically increasing sequence of natural numbers. Then we call the sequence (a_{n_k}) a **subsequence** of (a_n).

Examples: 1) The prime numbers are a subsequence of the positive integers.

2) $b_n = \dfrac{1}{1+n^2}$ is a subsequence of $a_n = \dfrac{1}{1+n}$ $(b_n = a_{n^2})$.

3) $c_n = \dfrac{1}{n!}$ is a subsequence of $a_n = \dfrac{1}{n}$ $(c_n = a_{n!})$.

Remark. A subsequence can be obtained from a given sequence by deleting some or no elements without changing the order of the remaining elements.
For example, 2, 4, 6, 8, ... is a subsequence of 1, 2, 3, 4, 5, 6, 7, 8, ...,
but 4, 2, 8, 6, ... is not a subsequence of it.

Remark. If (n_k) is a strictly monotonically increasing sequence of natural numbers, then $n_k \xrightarrow{k\to\infty} \infty$ since $n_k \geq n_1 + k - 1$.

Theorem. $\lim\limits_{n\to\infty} a_n = A$ if and only for all (a_{n_k}) subsequences $\lim\limits_{k\to\infty} a_{n_k} = A$.

Proof. 1) Assume that all (a_{n_k}) subsequences tend to the same limit A.
Since (a_{n+1}) is also a subsequence of (a_n), then $\lim\limits_{n\to\infty} a_{n+1} = A$,
so for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that if $n > N$ then $|a_{n+1} - A| < \varepsilon$.
Then obviously $|a_n - A| < \varepsilon$ if $n > N+1$, so $\lim\limits_{n\to\infty} a_n = A$ also holds.

2) Assume that $\lim\limits_{n\to\infty} a_n = A$ and let (a_{n_k}) be a subsequence of (a_n).
Then for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$, such that if $n > N$, then $|a_n - A| < \varepsilon$.
Since $n_k \xrightarrow{k\to\infty} \infty$, then for the number $N \in \mathbb{N}$ above, there exists $S \in \mathbb{N}$ such that if $k > S$, then $n_k > N$, so $|a_{n_k} - A| < \varepsilon$, therefore $a_{n_k} \xrightarrow{k\to\infty} A$.

The Sandwich Theorem and two applications

Theorem (Sandwich Theorem). If $a_n \xrightarrow{n\to\infty} A \in \mathbb{R}$, $c_n \xrightarrow{n\to\infty} A \in \mathbb{R}$ and $a_n \leq b_n \leq c_n$ for all $n > N$, then $b_n \xrightarrow{n\to\infty} A \in \mathbb{R}$

Proof. Let $\varepsilon > 0$ be fixed. Then
there exists $N_1 \in \mathbb{N}$ such that if $n > N_1$ then $A - \varepsilon < a_n < A + \varepsilon$ and
there exists $N_2 \in \mathbb{N}$ such that if $n > N_2$ then $A - \varepsilon < c_n < A + \varepsilon$.
So if $n > \max\{N, N_1, N_2\}$ then
$A - \varepsilon < a_n \leq b_n \leq c_n < A + \varepsilon \implies |b_n - A| < \varepsilon$.

Theorem. $\lim_{n\to\infty} \sqrt[n]{n} = 1.$

1st proof. Apply the AM-GM inequality for $a_1 = .. . a_{n-2} = 1,\ a_{n-1} = a_n = \sqrt{n}$. Then

$$1 \le \sqrt[n]{n} = \sqrt[n]{1 \cdot \ldots \cdot 1 \cdot \sqrt{n} \cdot \sqrt{n}} \le \frac{(n-2) + 2\sqrt{n}}{n} \le 1 + \frac{2}{\sqrt{n}} \longrightarrow 1 + 0 - 0 = 1,$$

so by the Sandwich Theorem, $\sqrt[n]{n} \longrightarrow 1$.

2nd proof. Since $\sqrt[n]{n} \ge 1$ then we can write $\sqrt[n]{n} = 1 + \delta_n$, where $\delta_n \ge 0$. Then by the binomial theorem, n can be estimated from below:

$$n = (1+\delta_n)^n = 1 + n\,\delta_n + \binom{n}{2}\delta_n^2 + \ldots + \binom{n}{n}\delta_n^2 \ge \binom{n}{2}\delta_n^2 = \frac{n(n-1)}{2}\delta_n^2, \text{ from where}$$

$$0 \le \delta_n \le \sqrt{\frac{2}{n-1}} \longrightarrow 0, \text{ so by the Sandwich Theorem, } \delta_n \longrightarrow 0 \text{ and thus } \sqrt[n]{n} \longrightarrow 1.$$

Theorem. If $p > 0$ then $\lim_{n\to\infty} \sqrt[n]{p} = 1.$

1st proof. Assume that $p \ge 1$ and apply the AM-GM inequality for $a_1 = .. . a_{n-2} = 1,\ a_{n-1} = a_n = \sqrt{p}$. Then

$$1 \le \sqrt[n]{p} = \sqrt[n]{1 \cdot \ldots \cdot 1 \cdot \sqrt{p} \cdot \sqrt{p}} \le \frac{(n-2) + 2\sqrt{p}}{n} \le 1 + \frac{2\sqrt{p}}{n} \longrightarrow 1 + 0 = 1,$$

so by the Sandwich Theorem, $\sqrt[n]{p} \longrightarrow 1$.

If $0 < p < 1$, then $\frac{1}{p} > 1$, so $\sqrt[n]{p} = \frac{1}{\sqrt[n]{\frac{1}{p}}} \longrightarrow 1$.

2nd proof. If $p \ge 1$ then $\sqrt[n]{p} \ge 1$, so we can write $\sqrt[n]{p} = 1 + \delta_n$, where $\delta_n \ge 0$. Then by the binomial theorem, n can be estimated from below:

$$p = (1+\delta_n)^n = 1 + n\,\delta_n + \binom{n}{2}\delta_n^2 + \ldots + \binom{n}{n}\delta_n^2 \ge n\,\delta_n,$$

from where $0 \le \delta_n \le \frac{p}{n} \longrightarrow 0$, so by the Sandwich Theorem, $\delta_n \longrightarrow 0$ and thus $\sqrt[n]{p} \longrightarrow 1$.

The case $0 < p < 1$ is the same as before.

3rd proof. If $p \ge 1$ then $\sqrt[n]{p} \ge 1$, so we can write $\sqrt[n]{p} = 1 + \delta_n$, where $\delta_n \ge 0$. We show that $\delta_n \longrightarrow 0$.

By the Bernoulli inequality $p = (1+\delta_n)^n \ge 1 + n\,\delta_n \implies \frac{p-1}{n} \ge \delta_n > 0$.

Since $\frac{p-1}{n} \longrightarrow 0$ then by the Sandwich Theorem $\delta_n \longrightarrow 0$, so $\sqrt[n]{p} \longrightarrow 1$.

The case $0 < p < 1$ is the same as before.

Examples

Exercise 1. Calculate the limit of $a_n = \sqrt[3n]{n}$.

1st solution: $a_n = \sqrt[3n]{n} = \sqrt[3n]{\dfrac{3n}{3}} = \dfrac{\sqrt[3n]{3n}}{\sqrt[3n]{3}} \longrightarrow \dfrac{1}{1} = 1$. Here we use that $\sqrt[3n]{3n} \longrightarrow 1$, since it is a subsequence of $\sqrt[n]{n}$, and similarly $\sqrt[3n]{3} \longrightarrow 1$, since it is a subsequence of $\sqrt[n]{3}$.

2nd solution: $a_n = \sqrt[3n]{n} = \sqrt[3]{\sqrt[n]{n}} \longrightarrow \sqrt[3]{1} = 1$.

3rd solution: Since $1 \le a_n = \sqrt[3n]{n} \le \sqrt[3n]{3n}$ for all $n \in \mathbb{N}$ and $\sqrt[3n]{3n} \longrightarrow 1$ then by the Sandwich Theorem, $a_n \longrightarrow 1$.

Exercise 2. Calculate the limit of $a_n = \sqrt[n]{\dfrac{2n^5 + 5n}{8n^2 - 2}}$.

Solution. Estimating a_n from above and from below:

$$a_n = \sqrt[n]{\frac{2n^5 + 5n}{8n^2 - 2}} \le \sqrt[n]{\frac{2n^5 + 5n^5}{8n^2 - 2n^2}} = \sqrt[n]{\frac{7n^5}{6n^2}} = \sqrt[n]{\frac{7}{6}} \cdot \left(\sqrt[n]{n}\right)^3 \longrightarrow 1 \cdot 1^3 = 1,$$

$$a_n = \sqrt[n]{\frac{2n^5 + 5n}{8n^2 - 2}} \ge \sqrt[n]{\frac{2n^5 + 0}{8n^2 + 0}} = \sqrt[n]{\frac{2n^5}{8n^2}} = \sqrt[n]{\frac{2}{8}} \cdot \left(\sqrt[n]{n}\right)^3 \longrightarrow 1 \cdot 1^3 = 1,$$

so by the Sandwich Theorem, $a_n \longrightarrow 1$.

Exercise 2. Calculate the limit of $a_n = \sqrt[n]{\dfrac{3^n + 5^n}{2^n + 4^n}}$.

Solution. Estimating a_n from above and from below:

$$a_n = \sqrt[n]{\frac{3^n + 5^n}{2^n + 4^n}} \le \sqrt[n]{\frac{5^n + 5^n}{0 + 4^n}} = \sqrt[n]{2} \cdot \frac{5}{4} \longrightarrow 1 \cdot \frac{5}{4} = \frac{5}{4},$$

$$a_n = \sqrt[n]{\frac{3^n + 5^n}{2^n + 4^n}} \ge \sqrt[n]{\frac{0 + 5^n}{4^n + 4^n}} = \sqrt[n]{\frac{1}{2}} \cdot \frac{5}{4} \longrightarrow 1 \cdot \frac{5}{4} = \frac{5}{4},$$

so by the Sandwich Theorem, $a_n \longrightarrow \frac{5}{4}$.

Monotonic sequences

Theorem. If (a_n) is monotonically increasing and not bounded above, then $a_n \overset{n\to\infty}{\longrightarrow} \infty$.

Proof. Let $P > 0$ be fixed. Since it is not an upper bound, there exists an $N \in \mathbb{N}$ such that $a_N > P$. By the monotonicity, if $n > N$ then $a_n \ge a_N > P$.

Consequence. If (a_n) is monotonically decreasing and not bounded below, then $a_n \xrightarrow{n\to\infty} -\infty$.

Theorem. (1) If (a_n) is monotonically increasing and bounded above, then (a_n) is convergent and $\lim\limits_{n\to\infty} a_n = \sup\{a_n : n \in \mathbb{N}\}$.

(2) If (a_n) is monotonically decreasing and bounded below, then (a_n) is convergent and $\lim\limits_{n\to\infty} a_n = \inf\{a_n : n \in \mathbb{N}\}$.

Consequence. If (a_n) is monotonic and bounded then (a_n) is convergent.

Proof of part (1).

1. Let $A = \sup\{a_k : k \in \mathbb{N}\}$, then $a_n \leq A$ for all $n \in \mathbb{N}$.
2. Assume indirectly that $\lim\limits_{n\to\infty} a_n \neq A$. Then there exists $\varepsilon > 0$, such that for all $N \in \mathbb{N}$ there exists $n > N$, such that $a_n \leq A - \varepsilon$.
3. By the monotonicity $a_N \leq a_n$, so $a_N \leq A - \varepsilon$ for all $N \in \mathbb{N}$.
4. However, this is a contradiction, since A is the smallest upper bound of the sequence, so $A - \varepsilon$ is not an upper bound.

Therefore for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that if $n > N$ then $A - \varepsilon < a_n \leq A < A + \varepsilon$, so $\lim\limits_{n\to\infty} a_n = A$.

Recursive sequences

In many cases, the convergence of recursively given sequences can be investigated by the application of the previous theorem.

Exercise 1. Let $0 < a < 1$ and $b_n = a^n$. Prove that the sequence (b_n) is convergent and find its limit.

Solution. Since $0 < b_{n+1} = a^{n+1} < a^n = b_n < 1$ then (b_n) is bounded and monotonically decreasing. So it is convergent, let $A = \lim\limits_{n\to\infty} b_n$. Then

$$A = \lim_{n\to\infty} b_{n+1} = \lim_{n\to\infty} a \cdot b_n = a \cdot A \iff A(1-a) = 0, \text{ so } A = 0.$$

Exercise 2. Let $a_1 = 4$ and $a_{n+1} = 8 - \dfrac{15}{a_n}$. Prove that the sequence (a_n) is convergent and find its limit.

Solution. The first few terms of the sequence:

$$a_1 = 4,\ a_2 = 4.25,\ a_3 = 4.47059,\ a_4 = 4.64474,\ a_5 = 4.77054,\ \ldots$$

1) First we calculate the possible limits of (a_n). If (a_n) is convergent then

$$A = \lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1} = 8 - \frac{15}{A} \implies A^2 - 8A + 15 = (A-3)(A-5) = 0, \text{ therefore } A = 3 \text{ or } A = 5.$$

2) Next, we investigate the boundedness and monotonicity of (a_n). If we prove that (a_n) is bounded and monotonically increasing or decreasing, then (a_n) is convergent and its limit is the supremum or the infimum of the sequence.

(i) First we prove boundedness by induction, that is, we prove that $3 < a_n < 5$ for all $n \in \mathbb{N}^+$.

I. The statement is true for $n = 1$: $3 < a_1 = 4 < 5$.

II. Assume that $3 < a_n < 5$. Then

$$3 < a_n < 5 \Longrightarrow \frac{1}{5} < \frac{1}{a_n} < \frac{1}{3} \Longrightarrow 3 < \frac{15}{a_n} < 5 \Longrightarrow -3 > -\frac{15}{a_n} > -5 \Longrightarrow 3 < 8 - \frac{15}{a_n} = a_{n+1} < 5.$$

(ii) Next we prove by induction that (a_n) is monotonically increasing, that is, $a_n < a_{n+1}$ for all $n \in \mathbb{N}^+$.

I. The statement is true for $n = 1$: $a_1 = 4 < a_2 = \frac{17}{4} = 4.25$

II. Assume that $a_n < a_{n+1}$. Then

$$a_n < a_{n+1} \Longrightarrow \frac{1}{a_n} > \frac{1}{a_{n+1}} \text{ (since } a_n > 3 > 0) \Longrightarrow -\frac{15}{a_n} < -\frac{15}{a_{n+1}} \Longrightarrow a_{n+1} = 8 - \frac{15}{a_n} < 8 - \frac{15}{a_{n+1}} = a_{n+2}.$$

Since (a_n) is monotonic increasing and bounded then (a_n) is convergent. The limit of (a_n) cannot be $A = 3$, since $a_1 = 4$ and the sequence is monotonic increasing. Therefore $\lim\limits_{n\to\infty} a_n = 5$.

The sequence $a_n = \left(1 + \frac{1}{n}\right)^n$

Theorem. The sequence $a_n = \left(1 + \frac{1}{n}\right)^n$ is monotonically increasing and bounded, so it is convergent.

1st proof. a) Monotonicity. We use the inequality between the arithmetic and geometric means:

if $a_1, a_2, \ldots, a_k \geq 0$ then $\sqrt[k]{a_1 a_2 \ldots a_k} \leq \dfrac{a_1 + a_2 + \ldots + a_k}{k}$.

Let $a_1 = \ldots = a_n = 1 + \frac{1}{n}$ and $a_{n+1} = 1$. Then $\sqrt[n+1]{\left(1 + \frac{1}{n}\right)^n \cdot 1} \leq \dfrac{n\left(1 + \frac{1}{n}\right) + 1}{n+1} = 1 + \dfrac{1}{n+1}$,

so $a_n = \left(1 + \frac{1}{n}\right)^n \leq \left(1 + \frac{1}{n+1}\right)^{n+1} = a_{n+1}$ for all $n \in \mathbb{N}$.

b) Boundedness. We use the inequality between the arithmetic and geometric means

for the numbers $a_1 = \ldots = a_n = 1 + \frac{1}{n}$ and $a_{n+1} = a_{n+2} = \frac{1}{2}$. Then

$$\sqrt[n+2]{\left(1 + \frac{1}{n}\right)^n \cdot \frac{1}{4}} \leq \frac{n\left(1 + \frac{1}{n}\right) + 2 \cdot \frac{1}{2}}{n+2} = 1, \quad \text{so } a_n = \left(1 + \frac{1}{n}\right)^n \leq 4 \text{ for all } n \in \mathbb{N}.$$

2nd proof with the binomial theorem

a) Boundedness. $a_n = \left(1 + \frac{1}{n}\right)^n = \sum\limits_{k=0}^{n} \binom{n}{k} \left(\frac{1}{n}\right)^k = 1 + 1 + \sum\limits_{k=2}^{n} \dfrac{n(n-1)\ldots(n-(k-1))}{k!} \cdot \dfrac{1}{n^k} =$

$$= 1 + 1 + \sum_{k=2}^{n} \frac{1}{k!} \cdot \frac{n}{n} \cdot \frac{n-1}{n} \cdot \ldots \cdot \frac{n-(k-1)}{n} < 1 + 1 + \sum_{k=2}^{n} \frac{1}{k!} \cdot 1 \cdot \ldots \cdot 1 = \sum_{k=0}^{n} \frac{1}{k!} := s_n.$$

The sequence (s_n) is bounded above since the terms can be estimated from above by the terms of a geometric sequence with ratio $\frac{1}{2}$:

$$s_n = 1 + 1 + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} + \ldots + \frac{1}{1 \cdot 2 \cdot \ldots \cdot n} < 1 + \left(1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots + \frac{1}{2^{n-1}}\right) \leq$$

$$\leq 1 + \frac{1-\left(\frac{1}{2}\right)^n}{1-\frac{1}{2}} = 3 - \left(\frac{1}{2}\right)^{n-1} < 3. \quad \text{So } a_n = \left(1+\frac{1}{n}\right)^n < s_n = \sum_{k=0}^{n} \frac{1}{k!} < 3.$$

b) Monotonicity.

$$a_{n+1} = \left(1+\frac{1}{n+1}\right)^{n+1} = \sum_{k=0}^{n+1} \binom{n+1}{k} \left(\frac{1}{n+1}\right)^k = 2 + \sum_{k=2}^{n+1} \frac{1}{k!} \cdot \frac{n+1}{n+1} \cdot \frac{n}{n+1} \cdot \frac{n-1}{n+1} \cdot \ldots \cdot \frac{(n+1)-(k-1)}{n+1} =$$

$$= 2 + \sum_{k=2}^{n+1} \frac{1}{k!} \left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right) \ldots \left(1-\frac{k-1}{n+1}\right) =$$

$$= 2 + \sum_{k=2}^{n} \frac{1}{k!} \left(\mathbf{1}-\frac{\mathbf{1}}{\mathbf{n+1}}\right) \ldots \left(\mathbf{1}-\frac{\mathbf{k-1}}{\mathbf{n+1}}\right) + \binom{\mathbf{n+1}}{\mathbf{n+1}} \frac{\mathbf{1}}{(\mathbf{n+1})^{\mathbf{n+1}}} >$$

$$> 2 + \sum_{k=2}^{n} \frac{1}{k!} \left(\mathbf{1}-\frac{\mathbf{1}}{\mathbf{n}}\right) \ldots \left(\mathbf{1}-\frac{\mathbf{k-1}}{\mathbf{n}}\right) + \mathbf{0} = a_n. \quad \text{So } a_n < a_{n+1}.$$

Definition: The number e is defined as the limit of the above sequence:

$$e := \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n.$$

Remark: From the 2nd proof it follows that $2 < e < 3$.

Some terms of the sequence are: $a_1 = 2$, $a_2 = 2.25$, $a_3 \approx 2.37$, $a_4 \approx 2.44$, $a_5 \approx 2.488$
$a_{10} \approx 2.59$, $a_{20} \approx 2.65$, $a_{100} \approx 2.70481$, $a_{200} \approx 2.71152$
$a_{1000} \approx 2.71692$, $a_{10\,000} \approx 2.71815$

Theorems. **1)** The number $e \approx 2.71828182845904523536028 7\ldots$ is irrational.

2) $\lim_{n\to\infty} \left(1+\frac{x}{n}\right)^n = e^x$ for all $x \in \mathbb{R}$

3) If $x_n \xrightarrow{n\to\infty} \infty$, then $\lim_{n\to\infty} \left(1+\frac{1}{x_n}\right)^{x_n} = e$.

4) $e = \lim_{n\to\infty} \left(1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\ldots+\frac{1}{n!}\right) = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{1}{k!} = \sum_{k=0}^{\infty} \frac{1}{k!}$

Remark. The convergence of the series $\sum_{n=0}^{\infty} \frac{1}{n!}$ is very fast, for example

$\sum_{n=0}^{6} \frac{1}{n!} = 1+1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+\frac{1}{6!} \approx 2.718\ldots$ (3 digits are accurate)

$\sum_{n=0}^{10} \frac{1}{n!} \approx 2.7182818\ldots$ (7 digits are accurate)

$\sum_{n=0}^{15} \frac{1}{n!} \approx 2.71828182845\ldots$ (11 digits are accurate)

$\sum_{n=0}^{20} \frac{1}{n!} \approx 2.7182818284590452353\ldots$ (19 digits are accurate)

Exercises

The sequence $a_n = \left(1 + \frac{1}{n}\right)^n$

1. $a_n = \left(1 + \dfrac{1}{n^3 + n + 6}\right)^{n^3+n+6} \longrightarrow e$, since it is a subsequence of $\left(1 + \dfrac{1}{n}\right)^n$.

2. $a_n = \left(1 + \dfrac{1}{n-6}\right)^n = \left(1 + \dfrac{1}{n-6}\right)^{n-6} \cdot \left(1 + \dfrac{1}{n-6}\right)^6 \longrightarrow e \cdot 1^6 = e$

3. $a_n = \left(1 + \dfrac{1}{6n+1}\right)^{6n-7} = \left(1 + \dfrac{1}{6n+1}\right)^{6n+1} \cdot \dfrac{1}{\left(1 + \dfrac{1}{6n+1}\right)^8} \longrightarrow e \cdot \dfrac{1}{1^8} = e$

4. $a_n = \left(\dfrac{n+3}{n+4}\right)^{n-2} = \left(\dfrac{n+4-1}{n+4}\right)^{n+4-6} = \left(1 + \dfrac{-1}{n+4}\right)^{n+4} \cdot \dfrac{1}{\left(\dfrac{n+4}{n+3}\right)^6} \longrightarrow e^{-1} \cdot \dfrac{1}{1^6} = \dfrac{1}{e}$

Here we used that $\dfrac{n+4}{n+3} = \dfrac{1 + \frac{4}{n}}{1 + \frac{3}{n}} \longrightarrow \dfrac{1+0}{1+0} = 1.$

Another solution: $a_n = \left(\dfrac{n+3}{n+4}\right)^{n-2} = \dfrac{\left(1 + \dfrac{3}{n}\right)^n}{\left(1 + \dfrac{4}{n}\right)^n} \cdot \left(\dfrac{n+4}{n+3}\right)^2 \longrightarrow \dfrac{e^3}{e^4} \cdot 1^2 = \dfrac{1}{e}$

5. $a_n = \left(\dfrac{n^2-2}{n^2+3}\right)^{n^2} = \dfrac{\left(1 + \dfrac{-2}{n^2}\right)^{n^2}}{\left(1 + \dfrac{3}{n^2}\right)^{n^2}} \longrightarrow \dfrac{e^{-2}}{e^3} = e^{-5}$

6. $a_n = \left(\dfrac{n+1}{n+6}\right)^{2n} = \left(\dfrac{\left(1 + \dfrac{1}{n}\right)^n}{\left(1 + \dfrac{6}{n}\right)^n}\right)^2 \longrightarrow \left(\dfrac{e^1}{e^6}\right)^2 = \left(e^{-5}\right)^2 = e^{-10}$

7. $a_n = \left(\dfrac{2n+2}{2n+9}\right)^{2n} = \dfrac{\left(1 + \dfrac{2}{2n}\right)^{2n}}{\left(1 + \dfrac{9}{2n}\right)^{2n}} \longrightarrow \dfrac{e^2}{e^9} = e^{-7}$

8. Calculate the limit of $a_n = \left(\dfrac{2n^2+5}{2n^2+3}\right)^{4n^2}$

1st solution. $a_n = \left(\dfrac{\left(1+\dfrac{5}{2n^2}\right)^{2n^2}}{\left(1+\dfrac{3}{2n^2}\right)^{2n^2}}\right)^2 \longrightarrow \left(\dfrac{e^5}{e^3}\right)^2 = e^4$

2nd solution. $a_n = \dfrac{\left(1+\dfrac{5\cdot 2}{4n^2}\right)^{4n^2}}{\left(1+\dfrac{3\cdot 2}{4n^2}\right)^{4n^2}} \longrightarrow \dfrac{e^{10}}{e^6} = e^4$

9. Calculate the limit of the following sequences:

$$a_n = \left(\frac{3n^2+1}{3n^2-2}\right)^{3n^2}, \qquad b_n = \left(\frac{3n^2+1}{3n^2-2}\right)^{9n^2}$$

$$c_n = \left(\frac{3n^2+1}{3n^2-2}\right)^{3n^3}, \qquad d_n = \left(\frac{3n^2+1}{3n^2-2}\right)^{3n}$$

Solution. $a_n = \left(\dfrac{3n^2+1}{3n^2-2}\right)^{3n^2} = \dfrac{\left(1+\dfrac{1}{3n^2}\right)^{3n^2}}{\left(1+\dfrac{-2}{3n^2}\right)^{3n^2}} \longrightarrow \dfrac{e}{e^{-2}} = e^3 = A$

$b_n = \left(\dfrac{3n^2+1}{3n^2-2}\right)^{9n^2} = (a_n)^3 \implies b_n \longrightarrow A^3 = e^9$

$c_n = \left(\dfrac{3n^2+1}{3n^2-2}\right)^{3n^3} = (a_n)^n$

In the estimation below we use that $2 < e < 3$, so $e^3 > 2^3 = 8$.

Since $a_n \longrightarrow e^3$ then $\exists\, N_1$ such that if $n > N_1$ then $c_n = (a_n)^n > 8^n \longrightarrow \infty \implies c_n \longrightarrow \infty$

$d_n = \left(\dfrac{3n^2+1}{3n^2-2}\right)^{3n} = \sqrt[n]{a_n}$

Since $a_n \longrightarrow e^3$ then for $\varepsilon = 0.1$ $\exists\, N_2$ such that if $n > N_2$ then $\sqrt[n]{e^3 - 0.1} \le d_n \le \sqrt[n]{e^3 + 0.1}$.

Since $\sqrt[n]{e^3 - 0.1} \longrightarrow 1$ and $\sqrt[n]{e^3 + 0.1} \longrightarrow 1$ then by the Sandwich Theorem $d_n \longrightarrow 1$.

Recursive sequences

1. Let $a_1 = \dfrac{4}{3}$ and $a_{n+1} = \dfrac{3 + a_n^2}{4}$, $n = 1, 2, \ldots$.

Prove that the sequence (a_n) is convergent and find its limit.

Solution. $a_1 \approx 1.33 > a_2 = \dfrac{3 + \left(\dfrac{4}{3}\right)^2}{4} \approx 1.194 > a_3 \approx 1.1067$

Conjecture: (a_n) is monotonically decreasing, so $a_n > a_{n+1} > 0$.

Proof: by induction.

I. $a_1 > a_2 > a_3 > 0$ is satisfied.
II. Assume that $a_{n-1} > a_n$. From the definition of the sequence it is obvious that $a_n > 0$ $\left(a_n = \frac{3 + a_{n-1}^2}{4} \geq \frac{3}{4} > 0\right)$. Then

$$a_{n-1} > a_n > 0 \implies a_{n-1}^2 > a_n^2 \implies 3 + a_{n-1}^2 > 3 + a_n^2 \implies a_n = \frac{3 + a_{n-1}^2}{4} > \frac{3 + a_n^2}{4} = a_{n+1}$$
$$\implies a_n > a_{n+1}.$$

Since (a_n) is monotonic decreasing and bounded below (since $a_n > 0$) then (a_n) is convergent, therefore $A = \lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{3 + a_n^2}{4}$

$$\implies A = \frac{3 + A^2}{4} \implies A^2 - 4A + 3 = (A-1)(A-3) = 0 \implies A = 1 \text{ or } A = 3.$$

Since $a_n < a_1 = \frac{4}{3}$ then $A = 3$ cannot be the case, so $A = \lim\limits_{n\to\infty} a_n = 1$.

2. Let $a_1 = 1$ and $a_{n+1} = \sqrt{6 + a_n}$, $n = 1, 2, \ldots$.
Is the sequence convergent? If so, what is the limit?

Solution. The first few terms of the sequence: $a_1 = 1$, $a_2 \approx 2.646$, $a_3 \approx 2.94$, ...
Since $\sqrt{6 + a_n} \geq 0$ then the terms of the sequence are positive.

1) First we calculate the possible limits of (a_n). If (a_n) is convergent then
$A = \lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} a_{n+1} = \lim\limits_{n\to\infty} \sqrt{6 + a_n} = \sqrt{6 + A} \implies A^2 - A - 6 = (A-3)(A+2) = 0$, from where $A = 3$ or $A = -2$. Since $a_n = \sqrt{6 + a_{n-1}} > 0$ then $A = -2$ cannot be the case, so the only possible limit is $A = 3$.

2) Next, we investigate the boundedness and monotonicity of (a_n).

(i) First we prove by induction that (a_n) is monotonically increasing and the terms are positive, that is, $0 < a_n < a_{n+1}$ for all $n \in \mathbb{N}^+$.
I. The statement is true for $n = 1$: $0 < a_1 = 1 < a_2 = \sqrt{7} \approx 2.646$
II. Assume that $0 < a_n < a_{n+1}$. Then

$$0 < a_n < a_{n+1} \implies 0 < 0 + 6 < 6 + a_n < 6 + a_{n+1} \implies 0 < \sqrt{6 + a_n} < \sqrt{6 + a_{n+1}} \implies 0 < a_{n+1} < a_{n+2}.$$

(ii) Next we prove that the sequence is bounded above. $A = 3$ is a suitable choice for the upper bound, that is, we show that $a_n < 3$ for all $n \in \mathbb{N}^+$.
I. The statement is true for $n = 1$: $a_1 = 1 < 3$
II. Assume that $a_n < 3$. Then $a_{n+1} = \sqrt{6 + a_n} < \sqrt{6 + 3} = 3$.

Since (a_n) is monotonic increasing and bounded above then (a_n) is convergent, so $\lim\limits_{n\to\infty} a_n = 3$.
We have seen that this is the only possible limit.

Remark. Monotonicity can also be proved as follows.

$$0 < a_n < a_{n+1} = \sqrt{6 + a_n} \iff a_n^2 < 6 + a_n \iff a_n^2 - a_n - 6 < 0 \iff -2 < a_n < 3.$$

Here $-2 < a_n$ trivially holds, since $a_n > 0$, and $a_n < 3$ can be proved by induction.

3. Let $a_1 = -3$ and $a_{n+1} = \dfrac{5 - 6\,a_n^2}{13}$, $n = 1, 2, \ldots$. Is the sequence convergent?

Solution. $a_1 = -3$, $a_2 \approx -3.769$, $a_3 \approx -6.1725$, ...

Is the sequence monotonic decreasing?

$$a_{n+1} = \frac{5 - 6\,a_n^2}{13} < a_n \iff 6\,a_n^2 + 13\,a_n - 5 > 0 \qquad \left(6\,x^2 + 13\,x - 5 = 0 \iff x_1 = -\frac{5}{2},\ x_2 = \frac{1}{3}\right)$$

It means that the sequence is monotonic decreasing if and only if $a_n < -\dfrac{5}{2}$ or $a_n > \dfrac{1}{3}$.

Homework: It can be proved by induction that $a_n \le -3 \left(< -\dfrac{5}{2}\right)$.

Therefore the sequence is monotonic decreasing with initial value $a_1 = -3$.

If the sequence were bounded from below then it would be convergent and for the limit we would have $A = \dfrac{5 - 6\,A^2}{13} \implies$ the possible values of A could be $A = -\dfrac{5}{2}$ or $A = \dfrac{1}{3}$.

Since $a_n \le -3$ for all n then these numbers cannot be the limit, so (a_n) is not convergent and therefore not bounded from below. Since (a_n) is monotonic decreasing then $\lim\limits_{n\to\infty} a_n = -\infty$.