Calculus 1

Number sequences, part 1.

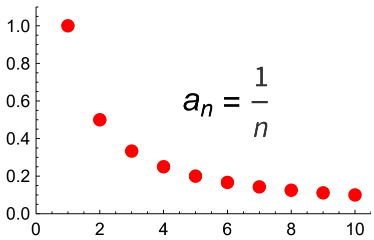
The concept and properties of sequences

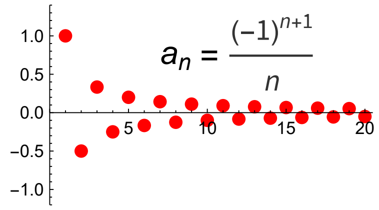
Definition: A number sequence is a function $f:\mathbb{N}\longrightarrow\mathbb{R}$ defined on the set of natural numbers.
Usual notation: $f(n)=a_n$ is the nth term of the sequence.
The notation of the sequence is (a_n) or $a_n,\ n=1,\ 2,\ \ldots$.

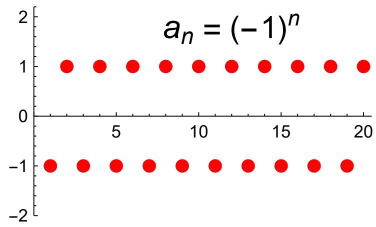
Remark: The function $f:\{k,\ k+1,\ k+2,\ \ldots\}\longrightarrow\mathbb{R}$ is also a sequence where $k=0,\ 1,\ 2,\ \ldots$.

Examples

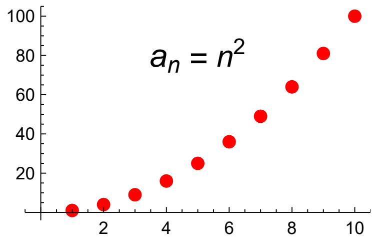
1) $1,\ \frac{1}{2},\ \frac{1}{3},\ \frac{1}{4},\ \ldots$ **2)** $1,\ -\frac{1}{2},\ \frac{1}{3},\ -\frac{1}{4},\ \ldots$ **3)** $1,\ -1,\ 1,\ -1,\ \ldots$

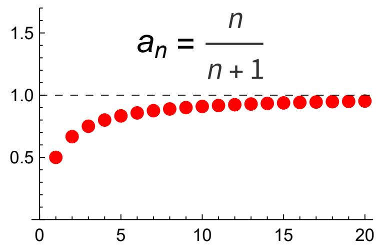


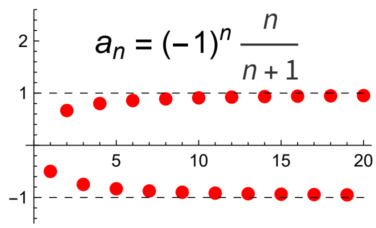




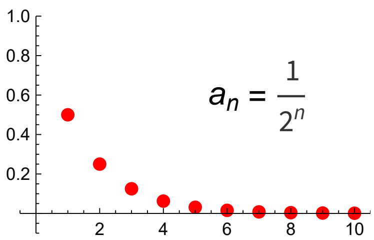
4) $1,\ 4,\ 9,\ 16,\ \ldots$ **5)** $\frac{1}{2},\ \frac{2}{3},\ \frac{3}{4},\ \frac{4}{5},\ \ldots$ **6)** $-\frac{1}{2},\ \frac{2}{3},\ -\frac{3}{4},\ \frac{4}{5},\ \ldots$

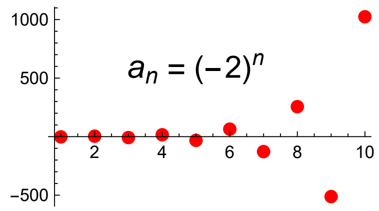


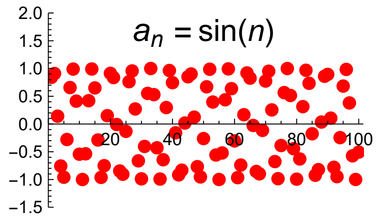




7) $\frac{1}{2},\ \frac{1}{3},\ \frac{1}{8},\ \frac{1}{16},\ \ldots$ **8)** $-2,\ 4,\ -8,\ 16,\ \ldots$ **9)** $\sin(1),\ \sin(2),\ \sin(3),\ \ldots$







Monotonicity

Definition:

The sequence (a_n) is $\begin{cases} \text{monotonically increasing,} \\ \text{strictly monotonically increasing,} \\ \text{monotonically decreasing,} \\ \text{strictly monotonically decreasing,} \end{cases}$ if for all $n \in \mathbb{N}$ $\begin{cases} a_n \leq a_{n+1} \\ a_n < a_{n+1} \\ a_n \geq a_{n+1} \\ a_n > a_{n+1} \end{cases}$.

Examples: Strictly monotonically decreasing: **1)** $a_n = \frac{1}{n}$, **7)** $a_n = \frac{1}{2^n}$

Strictly monotonically increasing: **4)** $a_n = n^2$, **5)** $a_n = \frac{n}{n+1}$

The other sequences are not monotonic.

Boundedness

Definition: The sequence (a_n) is
- bounded below, if there exists $A \in \mathbb{R}$ such that for all $n \in \mathbb{N}$: $A \leq a_n$.
- bounded above, if there exists $B \in \mathbb{R}$ such that for all $n \in \mathbb{N}$: $a_n \leq B$.
- bounded, if there exist $A \in \mathbb{R}$ and $B \in \mathbb{R}$ such that for all $n \in \mathbb{N}$: $A \leq a_n \leq B$.

Examples: Bounded sequences: **1)** $a_n = \frac{1}{n}$, **2)** $a_n = \frac{(-1)^n}{n}$, **3)** $a_n(-1)^n$, **5)** $a_n = \frac{n}{n+1}$,

6) $a_n = (-1)^n \frac{n}{n+1}$, **7)** $a_n = \frac{1}{2^n}$, **9)** $a_n = \sin(n)$

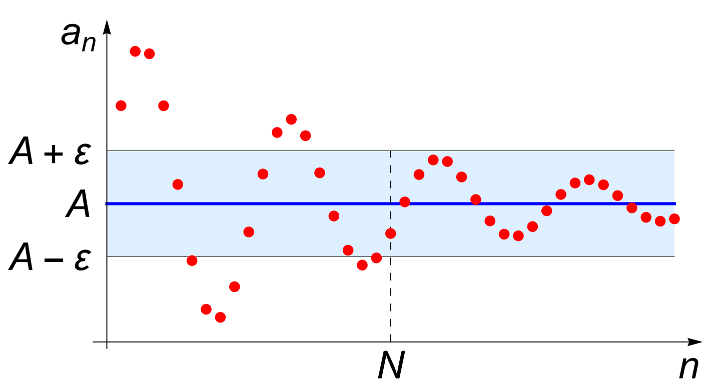
Convergent sequences

Definition: A sequence $(a_n) : \mathbb{N} \longrightarrow \mathbb{R}$ is **convergent**, and it tends to the limit $A \in \mathbb{R}$ if for all $\varepsilon > 0$ there exists a threshold index $N(\varepsilon) \in \mathbb{N}$ such that for all $n > N(\varepsilon)$, $|a_n - A| < \varepsilon$.

Notation: $\lim_{n\to\infty} a_n = A$ or $a_n \xrightarrow{n\to\infty} A$.

If a sequence if not convergent then it is **divergent**.

Remark: It is equivalent with the definition that for all $\varepsilon > 0$, the sequence has only finitely many terms outside of the interval $(A - \varepsilon, A + \varepsilon)$. (And the sequence has infinitely many terms in the interval.)



Examples for convergent sequences: **1)** $a_n = \frac{1}{n}$, **2)** $a_n = \frac{(-1)^n}{n}$, **5)** $a_n = \frac{n}{n+1}$, **7)** $a_n = \frac{1}{2^n}$

Exercises

1) Using the definition of the limit, show that a) $\lim_{n\to\infty} \frac{1}{n} = 0$ b) $\lim_{n\to\infty} \frac{(-1)^n}{n} = 0$.

Solution. Let $\varepsilon > 0$ be fixed. In both cases $|a_n - A| = \frac{1}{n} < \varepsilon \iff n > \frac{1}{\varepsilon}$

so with the choice $N(\varepsilon) \geq \left[\frac{1}{\varepsilon}\right]$ the definition holds.

For example, if $\varepsilon = 0.001$, then $N = 1000$ (or $N = 1500$ or $N = 2000$ etc.) is a suitable threshold index.

2) Using the definition of the limit, show that $\lim_{n\to\infty} \frac{6+n}{5.1-n} = -1$

Solution. Let $\varepsilon > 0$ be fixed. Then $|a_n - A| = \left|\frac{6+n}{5.1-n} - (-1)\right| = \left|\frac{11.1}{5.1-n}\right| \overset{\text{if } n>5}{=} \frac{11.1}{n-5.1} < \varepsilon \implies$

$n > 5.1 + \frac{11.1}{\varepsilon}$, so $N(\varepsilon) \geq \left[5.1 + \frac{11.1}{\varepsilon}\right]$.

3) Using the definition of the limit, show that $\lim_{n\to\infty} \frac{n^2-1}{2n^5+5n+8} = 0$

Solution. Let $\varepsilon > 0$ be fixed. Then $|a_n - A| = \left|\frac{n^2-1}{2n^5+5n+8}\right| = \frac{n^2-1}{2n^5+5n+8} < \varepsilon$.

This equation cannot be solved for n. However, it is not necessary to find the least possible threshold index, it is enough to show that a threshold index exists. So for the solution we use the transitive property of the inequalities, for example in the following way:

$$|a_n - A| = \left|\frac{n^2-1}{2n^5+5n+8}\right| = \frac{n^2-1}{2n^5+5n+8} < \frac{n^2-0}{2n^5+0+0} < \frac{1}{2n^3} < \varepsilon \iff n > \sqrt[3]{\frac{1}{2\varepsilon}}, \text{ so}$$

$N(\varepsilon) \geq \left[\sqrt[3]{\frac{1}{2\varepsilon}}\right]$.

Here we estimated the fraction from above in such a way that we increased the numerator and decreased the denominator.

4) Using the definition of the limit, show that $\lim_{n\to\infty} \frac{8n^4+3n+20}{2n^4-n^2+5} = 4$.

Solution. Let $\varepsilon > 0$ be fixed. Then $|a_n - A| = \left|\frac{8n^4+3n+20}{2n^4-n^2+5} - 4\right| = \left|\frac{4n^2+3n}{2n^4-n^2+5}\right| =$

$$= \frac{4n^2+3n}{2n^4-n^2+5} < \frac{4n^2+3n^2}{2n^4-n^4+0} = \frac{7}{n^2} < \varepsilon \iff n > \sqrt{\frac{7}{\varepsilon}}, \text{ so } N(\varepsilon) \geq \left[\sqrt{\frac{7}{\varepsilon}}\right].$$

Divergent sequences

If a sequence if not convergent then it is **divergent**.

Example: Show that $a_n = (-1)^n$ is divergent.

Solution. Since the terms of the sequence are $-1, 1, -1, 1, \ldots$ then the possible limits are only 1 and -1. We show that $A = 1$ is not the limit.
For example for $\varepsilon = 1$, the interval $(A - \varepsilon, A + \varepsilon) = (0, 2)$ contains infinitely many terms (the terms a_{2n}), however, there are infinitely many terms outside of this interval (the terms a_{2n-1}). It means that there is no suitable threshold index $N(\varepsilon)$ for $\varepsilon = 1$, so $A = 1$ is not the limit. Similarly, $A = -1$ is not the limit either, so the sequence is divergent.

Definition: The sequence $(a_n) : \mathbb{N} \longrightarrow \mathbb{R}$ tends to $+\infty$ if for all $P > 0$ there exists a threshold index $N(P) \in \mathbb{N}$ such that for all $n > N(P)$, $a_n > P$.

Notation: $\lim\limits_{n\to\infty} a_n = +\infty$ or $a_n \overset{n\to\infty}{\longrightarrow} +\infty$.

Definition: The sequence $(a_n) : \mathbb{N} \longrightarrow \mathbb{R}$ tends to $-\infty$ if for all $M < 0$ there exists a threshold index $N(M) \in \mathbb{N}$ such that for all $n > N(M)$, $a_n < M$.

Notation: $\lim\limits_{n\to\infty} a_n = -\infty$ or $a_n \overset{n\to\infty}{\longrightarrow} -\infty$.

Remark: $\lim\limits_{n\to\infty} a_n = -\infty$ if and only if $\lim\limits_{n\to\infty} (-a_n) = +\infty$.

Exercises

5) Let $a_n = 2n^3 + 3n + 5$. Show that $\lim\limits_{n\to\infty} a_n = \infty$.

Solution. Let $P > 0$ be fixed. Then $a_n = 2n^3 + 3n + 5 > 2n^3 > P \iff n > \sqrt[3]{\frac{P}{2}}$, so $N(P) \geq \left[\sqrt[3]{\frac{P}{2}}\right]$.

For example, if $P = 10^6$ then $N(P) = 80$ is a suitable threshold index.

6) Let $a_n = \dfrac{6 - n^2}{2 + n}$. Show that $\lim\limits_{n\to\infty} a_n = -\infty$.

Solution. We have to show that $a_n = \dfrac{6 - n^2}{2 + n} < M \; (< 0)$ if $n > N(M)$.

It is equivalent with the following condition: $-a_n = \dfrac{n^2 - 6}{n + 2} > -M \; (> 0)$ if $n > N(M)$.

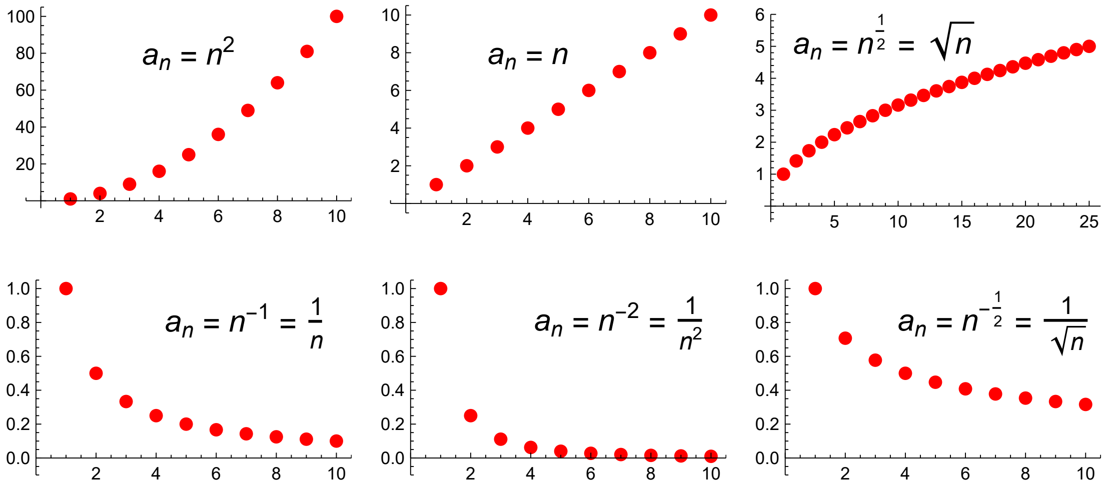
The exercise can be simplified with an estimation since we do not need to find the least possible threshold index: $\dfrac{n^2 - 6}{n + 2} > \dfrac{n^2 - \frac{n^2}{2}}{n + 2n} = \dfrac{n}{6} > -M \implies n > -6M$

In the estimation we used that $\frac{n^2}{2} > 6$ if $n \geq 4$. Therefore, $N(M) \geq \max\{4, [-6M]\}$ is a suitable threshold index.

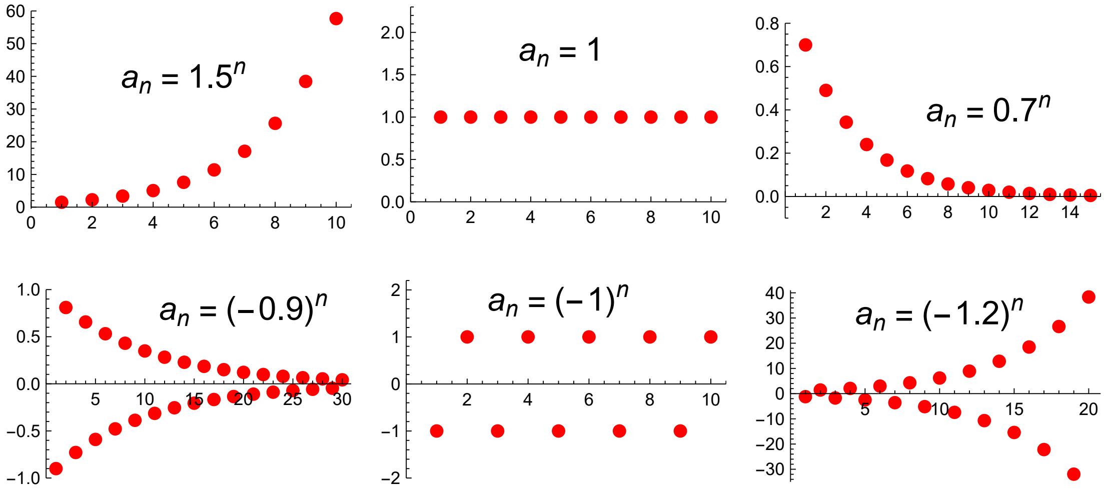
Examples

Using the above definitions, the following statements can easily be proved:

1) $\lim\limits_{n\to\infty} n^\alpha = \begin{cases} \infty, & \text{ha } \alpha > 0 \\ 1, & \text{ha } \alpha = 0 \\ 0, & \text{ha } \alpha < 0 \end{cases}$



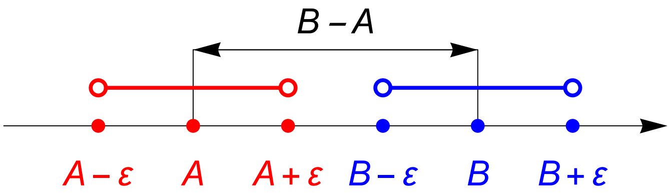
2) Limit of a geometric sequence: $\lim\limits_{n\to\infty} a^n = \begin{cases} \infty & \text{if } a > 1 \\ 1 & \text{if } a = 1 \\ 0 & \text{if } |a| < 1 \\ \text{does not exist} & \text{if } a \leq -1 \end{cases}$



Theorems about the limit

Theorem (uniqueness of the limit): If $\lim\limits_{n\to\infty} a_n = A$ and $\lim\limits_{n\to\infty} a_n = B$ then $A = B$.

Proof. We assume indirectly that $A \neq B$, for example $A < B$. Let $\varepsilon = \frac{B-A}{3} > 0$.



Since $a_n \longrightarrow A$ and $a_n \longrightarrow B$ then there exist threshold indexes $N_1 \in \mathbb{N}$ and $N_2 \in \mathbb{N}$ such that

- if $n > N_1$ then $A - \varepsilon < a_n < A + \varepsilon$ and
- if $n > N_2$ then $B - \varepsilon < a_n < B + \varepsilon$.

But in this case if $n > \max\{N_1, N_2\}$ then $a_n < A + \varepsilon < B - \varepsilon < a_n$. This is a contradiction, so $A = B$.

Theorem: If (a_n) is convergent, then it is bounded.

Proof. 1) Let $A = \lim\limits_{n\to\infty} a_n$. Then for $\varepsilon > 0$ there exists $N = N(\varepsilon) \in \mathbb{N}$ such that if $n > N$ then

$A - \varepsilon < a_n < A + \varepsilon$.

2) It means that the set $\{a_1, a_2, \ldots, a_N\}$ is finite, so the smallest element of $\{A - \varepsilon, a_1, \ldots, a_N\}$ is a lower bound and the largest element of $\{a_1, \ldots, a_N, A + \varepsilon\}$ is an upper bound of the set $\{a_n : n \in \mathbb{N}\}$.

3) Therefore for all n we have $\min\{A - \varepsilon, a_1, \ldots, a_N\} \leq a_n \leq \max\{a_1, \ldots, a_N, A + \varepsilon\}$.

Remark. Boundedness is a necessary but not sufficient condition for the convergence of a sequence. The converse of the statement is false, for example $a_n = (-1)^n$ is bounded but not convergent.

Example: Is the following sequence convergent or divergent? $a_n = \begin{cases} 2n+1, & \text{if } n \text{ is even} \\ \frac{1}{3n^2+1}, & \text{if } n \text{ is odd} \end{cases}$

Solution. The sequence is divergent, since it is not bounded. If $a_{2m} = 2 \cdot 2m + 1 = 4m + 1 \leq k \quad \forall m \in \mathbb{N}$ then it contradicts the Archimedian axiom.

Operations with convergent sequences

Theorem 1. If $a_n \xrightarrow{n\to\infty} A \in \mathbb{R}$ and $b_n \xrightarrow{n\to\infty} B \in \mathbb{R}$ then $a_n + b_n \xrightarrow{n\to\infty} A + B$. (Sum Rule)

Proof. Let $\varepsilon > 0$ be fixed. Since $a_n \xrightarrow{n\to\infty} A$ and $b_n \xrightarrow{n\to\infty} B$, then for $\frac{\varepsilon}{2}$ there exists $N_1 \in \mathbb{N}$ and $N_2 \in \mathbb{N}$ such that

- if $n > N_1$, then $\left| a_n - A \right| < \frac{\varepsilon}{2}$ and
- if $n > N_2$, then $\left| b_n - B \right| < \frac{\varepsilon}{2}$.

Thus, if $n > N = \max\{N_1, N_2\}$ then $\left| (a_n + b_n) - (A + B) \right| \leq \left| a_n - A \right| + \left| b_n - B \right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

Here we used the triangle inequality: $|a + b| \leq |a| + |b|$.

Theorem 2. If $a_n \overset{n\to\infty}{\longrightarrow} A \in \mathbb{R}$ and $c \in \mathbb{R}$ then $c\,a_n \overset{n\to\infty}{\longrightarrow} c\,A$. (Constant Multiple Rule)

Proof. Let $\varepsilon > 0$ be fixed.

(i) If $c = 0$ then the statement is trivial.

(ii) If $c \neq 0$ then because of the convergence of a_n, for $\frac{\varepsilon}{|c|}$ there exists $N \in \mathbb{N}$ such that if $n > N$ then $\left| a_n - A \right| < \frac{\varepsilon}{|c|}$. Thus, if $n > N$ then

$$|c\,a_n - c\,A| = |c\,(a_n - A)| = \left|c\right| \cdot \left|a_n - A\right| < \left|c\right| \cdot \frac{\varepsilon}{|c|} = \varepsilon.$$

Here we used that $|a\,b| = |a|\,|b|$.

Consequence. (i) If $a_n \overset{n\to\infty}{\longrightarrow} A \in \mathbb{R}$ then $-a_n \overset{n\to\infty}{\longrightarrow} -A$. (Here $c = -1$.)

(ii) If $a_n \overset{n\to\infty}{\longrightarrow} A \in \mathbb{R}$ and $b_n \overset{n\to\infty}{\longrightarrow} B \in \mathbb{R}$ then

$$a_n - b_n = a_n + (-b_n) \overset{n\to\infty}{\longrightarrow} A + (-B) = A - B. \text{ (Difference Rule)}$$

Theorem 3. (i) If $a_n \overset{n\to\infty}{\longrightarrow} 0$ and $b_n \overset{n\to\infty}{\longrightarrow} 0$ then $a_n\,b_n \overset{n\to\infty}{\longrightarrow} 0$.

(ii) If $a_n \overset{n\to\infty}{\longrightarrow} A \in \mathbb{R}$ and $b_n \overset{n\to\infty}{\longrightarrow} B \in \mathbb{R}$ then $a_n\,b_n \overset{n\to\infty}{\longrightarrow} A\,B$. (Product Rule)

Proof. Let $\varepsilon > 0$ be fixed.

(i) Since $a_n \overset{n\to\infty}{\longrightarrow} 0$ and $b_n \overset{n\to\infty}{\longrightarrow} 0$, then

- for $\frac{\varepsilon}{2}$ there exists $N_1 \in \mathbb{N}$ such that if $n > N_1$ then $\left|a_n - 0\right| < \frac{\varepsilon}{2}$ and
- for 2 there exists $N_2 \in \mathbb{N}$ such that if $n > N_2$ then $|b_n - 0| < 2$.

Thus, if $n > N = \max\{N_1, N_2\}$ then $|a_n\,b_n - 0| = \left|a_n\right| \cdot \left|b_n\right| < \frac{\varepsilon}{2} \cdot 2 = \varepsilon$.

(ii) It is obvious that if $c_n \equiv A$ for all $n \in \mathbb{N}$ (constant sequence) then $c_n \overset{n\to\infty}{\longrightarrow} A$.

Thus $a_n - A \overset{n\to\infty}{\longrightarrow} A - A = 0$ and $b_n - B \overset{n\to\infty}{\longrightarrow} B - B = 0$.

Applying part (i) we get that $(a_n - A)\,(b_n - B) \overset{n\to\infty}{\longrightarrow} 0$, that is,

$$a_n\,b_n - A\,b_n - B\,a_n + A\,B \overset{n\to\infty}{\longrightarrow} 0.$$

Then

$$a_n\,b_n = (a_n\,b_n - A\,b_n - B\,a_n + A\,B) + (A\,b_n + B\,a_n - A\,B) \overset{n\to\infty}{\longrightarrow} 0 + (A\,B + A\,B - A\,B) = A\,B.$$

Theorem 4. If $a_n \overset{n\to\infty}{\longrightarrow} 0$ and (b_n) is bounded then $a_n\,b_n \overset{n\to\infty}{\longrightarrow} 0$.

Proof. Let $\varepsilon > 0$ be fixed.

Since (b_n) is bounded then there exists $K > 0$ such that $|b_n| < K$ for all $n \in \mathbb{N}$.

Since $a_n \overset{n\to\infty}{\longrightarrow} 0$ then for $\frac{\varepsilon}{K}$ there exists $N \in \mathbb{N}$ such that if $n > N$ then $|a_n - 0| = \left|a_n\right| < \frac{\varepsilon}{K}$.

Thus, if $n > N$ then $|a_n\,b_n - 0| = \left|a_n\right| \cdot \left|b_n\right| < \frac{\varepsilon}{K} \cdot K = \varepsilon$.

Theorem 5. If $a_n \xrightarrow{n\to\infty} A \in \mathbb{R}$ then $|a_n| \xrightarrow{n\to\infty} |A|$.

Proof. $\big|\,|a_n| - |A|\,\big| \leq |a_n - A| < \varepsilon$ if $n > N(\varepsilon)$.

Remark. The converse of the statement is not true.

For example, $a_n = (-1)^n$ is divergent but $|a_n| = 1^n = 1 \xrightarrow{n\to\infty} 1$.

However, the following statement is true: $|a_n| \xrightarrow{n\to\infty} 0 \implies a_n \xrightarrow{n\to\infty} 0$.

Since $\big|\,|a_n| - 0\,\big| = |a_n| = |a_n - 0| < \varepsilon$ if $n > N(\varepsilon)$.

Theorem 6. (i) If $b_n \xrightarrow{n\to\infty} B \neq 0$ then $\frac{1}{b_n} \xrightarrow{n\to\infty} \frac{1}{B}$.

(ii) If $a_n \xrightarrow{n\to\infty} A \in \mathbb{R}$ and $b_n \xrightarrow{n\to\infty} B \neq 0$ then $\frac{a_n}{b_n} \xrightarrow{n\to\infty} \frac{A}{B}$. (Quotient Rule)

Proof. (i) First, by the convergence of (b_n) and by Theorem 5, $|b_n| \xrightarrow{n\to\infty} |B| \neq 0$ and thus

there exists $N_1 = N_1\left(\frac{|B|}{2}\right) \in \mathbb{N}$ such that if $n > N_1$ then

$$\big|\,|b_n| - |B|\,\big| < \frac{|B|}{2} \iff |B| - \frac{|B|}{2} < |b_n| < |B| + \frac{|B|}{2}.$$

Then $|b_n| > \frac{|B|}{2}$ for all $n > N_1$.

Second, for a fixed $\varepsilon > 0$ there exists $N_2 = N_2\left(\frac{|B|^2 \varepsilon}{2}\right) \in \mathbb{N}$ such that

if $n > N_2$ then $|b_n - B| < \frac{|B|^2 \varepsilon}{2}$. Therefore, if $n > N = \max\{N_1, N_2\}$ then

$$\left|\frac{1}{b_n} - \frac{1}{B}\right| = \left|\frac{B - b_n}{B \cdot b_n}\right| = \frac{|B - b_n|}{|B| \cdot |b_n|} < \frac{1}{|B| \cdot \frac{|B|}{2}} \cdot \frac{|B|^2 \varepsilon}{2} = \varepsilon.$$

(ii) By Theorem 3 and Theorem 6, part (i): $\frac{a_n}{b_n} = a_n \cdot \frac{1}{b_n} \xrightarrow{n\to\infty} A \cdot \frac{1}{B} = \frac{A}{B}$

Remark. By induction it can be proved that Theorem 1 and Theorem 3 can be generalized to the sum and product of **finitely many** convergent sequences. However, they are not true for infinitely many terms, as the following examples show.

Example. $\lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^{10} = 1^{10} = 1$ or $\lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^{k} = 1^k = 1$, where $k \in \mathbb{N}^+$ is a fixed constant, independent of n. However, $\lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^{n} \neq 1^n = 1$. Later we will see that $\lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^{n} = e$.

Example. $a_n = \frac{1}{n^2} + \frac{2}{n^2} + \ldots + \frac{500}{n^2} \longrightarrow 0 + 0 + \ldots + 0 = 0$

The number of the terms is 500 which is independent of n and thus applying Theorem 1 finitely many times, the correct result is 0.

Example. $b_n = \frac{1}{n^2} + \frac{2}{n^2} + \ldots + \frac{n}{n^2} \longrightarrow 0 + 0 + \ldots + 0 = 0$ is a WRONG SOLUTION!

Since $b_1 = \frac{1}{1^2}$, $b_2 = \frac{1}{2^2} + \frac{2}{2^2}$, $b_3 = \frac{1}{3^2} + \frac{2}{3^2} + \frac{3}{3^2}$, $b_4 = \frac{1}{4^2} + \frac{2}{4^2} + \frac{3}{4^2} + \frac{4}{4^2}, \ldots,$

then it can be seen that the number of the terms depends on n,
so b_n is not the sum of finitely many sequences and thus Theorem 1 cannot be generalized to this case. The correct solution is:

$$b_n = \frac{1 + 2 + \ldots + n}{n^2} = \frac{(1 + n) \cdot \frac{n}{2}}{n^2} = \frac{1 + n}{2 n} = \frac{\frac{1}{n} + 1}{2} \longrightarrow \frac{0 + 1}{2} = \frac{1}{2}$$

Example. $a_n = \frac{8 n^2 - n + 3}{n^2 + 9} = \frac{n^2}{n^2} \cdot \frac{8 - \frac{1}{n} + \frac{3}{n^2}}{1 + \frac{9}{n^2}} \longrightarrow 1 \cdot \frac{8 - 0 + 0}{1 + 0} = 8$

Example. Calculate the limit of $a_n = \left(\frac{2 n + 1}{3 - n}\right)^3 \cdot \frac{3 n^2 + 2 n}{2 + 6 n^2}$.

Solution. $a_n = \left(\frac{2 n}{-n}\right)^3 \cdot \left(\frac{1 + \frac{1}{2 n}}{1 - \frac{3}{n}}\right)^3 \cdot \frac{3 n^2}{6 n^2} \cdot \frac{1 + \frac{2}{3 n}}{1 + \frac{1}{3 n^2}} \longrightarrow -8 \cdot 1^3 \cdot \frac{1}{2} \cdot 1 = -4$

Here the product rule is used for the power.

Example. Calculate the limit of $a_n = \frac{n^2 - 5}{2 n^3 + 6 n} \cdot \sin\left(n^4 + 5 n + 8\right)$.

Solution. $a_n \longrightarrow 0$, since $b_n = \frac{n^2 - 5}{2 n^3 + 6 n} = \frac{n^2}{2 n^3} \cdot \frac{1 - \frac{5}{n^2}}{1 + \frac{3}{n^2}} \longrightarrow 0 \cdot 1$ and $c_n = \sin\left(n^4 + 5 n + 8\right)$ is bounded.

Example. $a_n = \frac{2^{2 n} + \cos\left(n^2\right)}{4^{n+1} - 5} = \frac{4^n}{4^n} \cdot \frac{1 + \left(\frac{1}{4}\right)^n \cdot \cos\left(n^2\right)}{4 - 5 \cdot \left(\frac{1}{4}\right)^n} \longrightarrow \frac{1 + 0}{4 - 0} = \frac{1}{4}$

Theorem 7. If $a_n \geq 0$ and $a_n \overset{n\to\infty}{\longrightarrow} A \geq 0$ then $\sqrt{a_n} \overset{n\to\infty}{\longrightarrow} \sqrt{A}$.

Proof. Let $\varepsilon > 0$ be fixed.

(i) If $a_n \overset{n\to\infty}{\longrightarrow} A = 0$ then there exists $N_1 = N_1(\varepsilon^2) \in \mathbb{N}$ such that if $n > N_1$ then $| a_n - 0 | = a_n < \varepsilon^2$.

Therefore, if $n > N_1$ then $\left| \sqrt{a_n} - 0 \right| = \sqrt{a_n} < \varepsilon$.

(ii) If $a_n \overset{n\to\infty}{\longrightarrow} A > 0$ then there exists $N_2 = N_2\left(\varepsilon \sqrt{A}\right) \in \mathbb{N}$ such that if $n > N_2$ then $\left| a_n - A \right| < \varepsilon \sqrt{A}$.

Therefore, if $n > N_2$ then

$$\left| \sqrt{a_n} - \sqrt{A} \right| = \left| \frac{a_n - A}{\sqrt{a_n} + \sqrt{A}} \right| = \frac{| a_n - A |}{\sqrt{a_n} + \sqrt{A}} \leq \frac{| a_n - A |}{0 + \sqrt{A}} < \frac{\varepsilon \sqrt{A}}{\sqrt{A}} = \varepsilon.$$

Remark. If $a_n \overset{n\to\infty}{\longrightarrow} A \geq 0$ then $\sqrt[k]{a_n} \overset{n\to\infty}{\longrightarrow} \sqrt[k]{A}$ for all $k \in \mathbb{N}^+$.

It can be proved by using the following identity: $a^k - b^k = (a - b)\left(a^{k-1} + a^{k-2}\, b + \ldots + a\, b^{k-1} + b^{k-1}\right)$.

Example. Calculate the limit of $a_n = \sqrt{4 n^2 + 5 n - 1} - \sqrt{4 n^2 + n + 3}$ (it has the form $\infty - \infty$)

Solution. $a_n = \alpha - \beta = \dfrac{(\alpha - \beta)(\alpha + \beta)}{\alpha + \beta} = \dfrac{\left(4 n^2 + 5 n - 1\right) - \left(4 n^2 + n + 3\right)}{\sqrt{4 n^2 + 5 n - 1} + \sqrt{4 n^2 + n + 3}} =$

$$= \frac{4 n - 4}{\sqrt{4 n^2 + 5 n - 1} + \sqrt{4 n^2 + n + 3}} = \frac{4 n}{\sqrt{4 n^2}} \frac{1 - \frac{1}{n}}{\sqrt{1 + \frac{5}{4 n} - \frac{1}{4 n^2}} + \sqrt{1 + \frac{1}{4 n} + \frac{3}{4 n^2}}} \longrightarrow$$

$$\longrightarrow 2 \cdot \frac{1 - 0}{\sqrt{1 + 0 - 0} + \sqrt{1 + 0 + 0}} = 1.$$

Additional theorems about the limit

Theorem. If $a_n \overset{n\to\infty}{\longrightarrow} \infty$ then $\dfrac{1}{a_n} \overset{n\to\infty}{\longrightarrow} 0$.

Proof. Let $\varepsilon > 0$ be fixed. Since $a_n \overset{n\to\infty}{\longrightarrow} \infty$, then for $P = \dfrac{1}{\varepsilon}$ there exists $N \in \mathbb{N}$ such that

if $n > N$ then $a_n > \dfrac{1}{\varepsilon} > 0$, so $\left| \dfrac{1}{a_n} - 0 \right| = \dfrac{1}{a_n} < \varepsilon$.

Question: Is it true that if $a_n \overset{n\to\infty}{\longrightarrow} 0$ then $\dfrac{1}{a_n} \overset{n\to\infty}{\longrightarrow} \infty$?

Answer: No, for example, if $a_n = -\dfrac{2}{n} \longrightarrow 0$ then $\dfrac{1}{a_n} = -\dfrac{n}{2} \longrightarrow -\infty$.

Or, if $a_n = \left(-\dfrac{1}{2}\right)^n \longrightarrow 0$ then for $b_n = \dfrac{1}{a_n} = (-2)^n$, $b_{2k} \longrightarrow \infty$ and $b_{2k} \longrightarrow -\infty$, so $\lim\limits_{n\to\infty} \dfrac{1}{a_n} \neq \infty$.

However, the following statements hold.

Theorem. a) If $a_n > 0$ and $a_n \overset{n\to\infty}{\longrightarrow} 0$ then $\frac{1}{a_n} \overset{n\to\infty}{\longrightarrow} \infty$. Notation: $\frac{1}{0+} \longrightarrow +\infty$.

b) If $a_n < 0$ and $a_n \overset{n\to\infty}{\longrightarrow} 0$ then $\frac{1}{a_n} \overset{n\to\infty}{\longrightarrow} -\infty$. Notation: $\frac{1}{0-} \longrightarrow -\infty$.

c) If $a_n \overset{n\to\infty}{\longrightarrow} 0$ then $\frac{1}{|\, a_n \,|} \overset{n\to\infty}{\longrightarrow} \infty$.

Proof. a) Let $P > 0$ be fixed. Since $0 < a_n \overset{n\to\infty}{\longrightarrow} 0$, then for $\varepsilon = \frac{1}{P}$ there exists $N \in \mathbb{N}$ such that

if $n > N$ then $a_n = \left| a_n - 0 \right| < \frac{1}{P}$, so $\frac{1}{a_n} > P$.

b), c): homework.

Theorem. If $a_n \overset{n\to\infty}{\longrightarrow} \infty$ and $b_n \geq a_n$ for $n > N$, then $b_n \longrightarrow \infty$.

Proof. Let $P > 0$ be fixed. Since $a_n \overset{n\to\infty}{\longrightarrow} \infty$, then there exists $N_1 \in \mathbb{N}$ such that if $n > N_1$ then $a_n > P$. So if $n > \max\{N, N_1\}$ then $b_n > P$.

Consequence. Suppose that $a_n \overset{n\to\infty}{\longrightarrow} \infty$, $b_n \overset{n\to\infty}{\longrightarrow} \infty$, $c_n \overset{n\to\infty}{\longrightarrow} c > 0$ and $|\, d_n \,| \leq K$ for all $n > \in \mathbb{N}$. Then

a) $a_n + b_n \overset{n\to\infty}{\longrightarrow} \infty$ b) $a_n \cdot b_n \overset{n\to\infty}{\longrightarrow} \infty$

c) $c_n \cdot a_n \overset{n\to\infty}{\longrightarrow} \infty$ d) $a_n + d_n \overset{n\to\infty}{\longrightarrow} \infty$

Proof. a) Since $a_n \overset{n\to\infty}{\longrightarrow} \infty$, it may be assumed that there exists $N \in \mathbb{N}$ such that $a_n \geq 0$ for $n > N$.

Then $a_n + b_n \geq b_n \overset{n\to\infty}{\longrightarrow} \infty$, so $a_n + b_n \overset{n\to\infty}{\longrightarrow} \infty$.

b) Since $a_n \overset{n\to\infty}{\longrightarrow} \infty$ and $b_n \overset{n\to\infty}{\longrightarrow} \infty$, it may be assumed that there exists $N \in \mathbb{N}$ such that $a_n \geq 1$ and $b_n \geq 0$ for $n > N$. Then $a_n \cdot b_n \geq b_n \overset{n\to\infty}{\longrightarrow} \infty$, so $a_n \cdot b_n \overset{n\to\infty}{\longrightarrow} \infty$.

c) Let $P > 0$ be fixed.

- Since $c_n \overset{n\to\infty}{\longrightarrow} c > 0$ then there exists $N_1 = N_1\left(\frac{c}{2}\right) \in \mathbb{N}$ such that $c_n > \frac{c}{2}$ if $n > N_1$.
- Since $a_n \overset{n\to\infty}{\longrightarrow} \infty$ then there exists $N_2 = N_2\left(\frac{2P}{c}\right) \in \mathbb{N}$ such that $a_n > \frac{2P}{c}$ if $n > N_2$.

So if $n > \max\{N_1, N_2\}$ then $c_n \cdot a_n > \frac{2P}{c} \cdot \frac{c}{2} = P$.

d) Let $P > 0$ be fixed. $a_n + d_n \geq a_n - K > P$ if and only if $a_n > K + P$.

Since $a_n \overset{n\to\infty}{\longrightarrow} \infty$ then for $K + P$ there exists $N \in \mathbb{N}$ such that $a_n > K + P$ if $n > N$.

Then for $n > N$, $a_n + d_n > P$ also holds, so $a_n + d_n \overset{n\to\infty}{\longrightarrow} \infty$.

Example. $a_n = 5n^2 + 2^n \cdot n - (-1)^n \overset{n\to\infty}{\longrightarrow} \infty$.

Remark. The above statements can be denoted in the following way:

a) $\infty+\infty \longrightarrow \infty$ b) $\infty\cdot\infty \longrightarrow \infty$

c) $c\cdot\infty \longrightarrow \infty$ (where $c>0$) d) ∞ + bounded $\longrightarrow \infty$.

Similar statements can be proved, for example,

$$\frac{0}{\infty}\longrightarrow 0,\quad \frac{\text{bounded}}{\infty}\to 0,\quad \frac{\infty}{+0}\to\infty,\quad \frac{\infty}{-0}\to-\infty.$$

The meaning of $\frac{0}{\infty}\longrightarrow 0$ is that if $a_n \overset{n\to\infty}{\longrightarrow} 0$ and $b_n \overset{n\to\infty}{\longrightarrow} \infty$ then $\frac{a_n}{b_n}\longrightarrow 0$.

Undefined forms: $\infty-\infty,\ 0\cdot\infty,\ \frac{\infty}{\infty},\ \frac{0}{0},\ 1^\infty,\ \infty^0,\ 0^0$

Examples for undefined forms:

1) Limit of the form $\infty-\infty$:

$a_n=n^2,\quad b_n=n,\quad a_n-b_n=n^2-n\to\infty$

$a_n=n,\quad b_n=n,\quad a_n-b_n=n-n=0\to 0$

$a_n=n,\quad b_n=n^2,\quad a_n-b_n=n-n^2\to-\infty$

2) Limit of the form $0\cdot\infty$:

$\frac{1}{n}\cdot n^2=n\to\infty,\quad \frac{1}{n}\cdot n=1\to 1,\quad \frac{1}{n^2}\cdot n=\frac{1}{n}\to 0,\quad \frac{(-1)^n}{n}\cdot n=(-1)^n$ (it doesn't have a limit)

3) Limit of the form $\frac{\infty}{\infty}$: $\frac{n}{n^2}=\frac{1}{n}\to 0,\quad \frac{n^2}{n}=n\to\infty,\quad \frac{n^2}{n^2}=1\to 1$

4) Limit of the form $\frac{0}{0}$:

$\frac{\frac{1}{n}}{\frac{1}{n^2}}=n\to\infty,\quad \frac{\frac{1}{n^2}}{\frac{1}{n}}=\frac{1}{n}\to 0,\quad \frac{\frac{1}{n}}{\frac{1}{n}}=1\to 1,\quad \frac{(-1)^n\frac{1}{n}}{\frac{1}{n^2}}=(-1)^n\cdot n$ (it doesn't have a limit)

Such statements are summarized in the following tables where $\overline{\mathbb{R}}=\mathbb{R}\cup\{\infty,-\infty\}$ denotes the extended set of real numbers. The meaning of $|\ .\ |$ is that $\lim\limits_{n\to\infty}\left|\frac{a_n}{b_n}\right|=\infty$.

Addition:

$\lim(a_n)$	$\lim(b_n)$	$\lim(a_n+b_n)$
$a \in \mathbb{R}$	$b \in \mathbb{R}$	$a+b$
∞	$b \in \mathbb{R}$	∞
$-\infty$	$b \in \mathbb{R}$	$-\infty$
∞	∞	∞
$-\infty$	$-\infty$	$-\infty$
∞	$-\infty$	?

Subtraction:

$\lim(a_n)$	$\lim(b_n)$	$\lim(a_n-b_n)$
$a \in \mathbb{R}$	$b \in \mathbb{R}$	$a-b$
∞	$b \in \mathbb{R}$	∞
$-\infty$	$b \in \mathbb{R}$	$-\infty$
∞	$-\infty$	∞
∞	∞	?
$-\infty$	$-\infty$	?

Multiplication:

$\lim(a_n)$	$\lim(b_n)$	$\lim(a_n b_n)$
$a \in \mathbb{R}$	$b \in \mathbb{R}$	$a\,b$
∞	$b>0$	∞
∞	$b<0$	$-\infty$
$-\infty$	$b>0$	$-\infty$
$-\infty$	$b<0$	∞
∞	∞	∞
∞	$-\infty$	$-\infty$
$-\infty$	$-\infty$	∞
∞	0	?
$-\infty$	0	?

Division:

$\lim(a_n)$	$\lim(b_n)$	$\lim(a_n/b_n)$
$a \in \mathbb{R}$	$b \in \mathbb{R} \setminus \{0\}$	$a\,/\,b$
∞	$b>0$	∞
∞	$b<0$	$-\infty$
$-\infty$	$b>0$	$-\infty$
$-\infty$	$b<0$	∞
$a \in \mathbb{R}$	$\pm\infty$	0
0	$b \in \overline{\mathbb{R}},\ b \neq 0$	0
$a \in \overline{\mathbb{R}},\ a \neq 0$	0	$\lvert\cdot\rvert = \infty$
0	0	?
$\pm\infty$	$\pm\infty$	?

Exercises

1) Calculate the limit of $a_n = \dfrac{3n^5+n^2-n}{n^3+3}$.

Solution. $a_n = \dfrac{3n^5+n^2-n}{n^3+3} > \dfrac{3n^5+0-n^5}{n^3+3n^3} = \dfrac{n^2}{2} \longrightarrow \infty \implies a_n \longrightarrow \infty$

or:

$$a_n = \frac{3n^5+n^2-n}{n^3+3} \geq \frac{n^5}{n^3} \cdot \frac{3+\frac{1}{n^3}-\frac{1}{n^4}}{1+\frac{3}{n^3}} \longrightarrow \infty,$$

since $b_n = \dfrac{n^5}{n^3} = n^2 \longrightarrow \infty$ and $c_n = \dfrac{3+\frac{1}{n^3}-\frac{1}{n^4}}{1+\frac{3}{n^3}} \longrightarrow \dfrac{3+0-0}{1+0} = 3 > 0.$

2) Calculate the limit of $a_n = \dfrac{3^{2n}}{4^n+3^{n+1}}$.

Solution. $a_n = \dfrac{3^{2n}}{4^n+3^{n+1}} = \left(\dfrac{9}{4}\right)^n \cdot \dfrac{1}{1+3\cdot\left(\frac{3}{4}\right)^n} > \left(\dfrac{9}{4}\right)^n \cdot \dfrac{1}{1+3\cdot 1} \longrightarrow \infty \implies a_n \longrightarrow \infty$

or:

$$a_n = a_n = \frac{3^{2n}}{4^n+3^{n+1}} = \left(\frac{9}{4}\right)^n \cdot \frac{1}{1+3\cdot\left(\frac{3}{4}\right)^n} \longrightarrow \infty,$$

since $b_n = \left(\frac{9}{4}\right)^n \longrightarrow \infty$ and $c_n = \frac{1}{1 + 3 \cdot \left(\frac{3}{4}\right)^n} \longrightarrow \frac{1}{1 + 3 \cdot 0} = 1 > 0.$

3) Calculate the limit of $a_n = \frac{2^{2n} + (-3)^{n-1}}{5^{n+2} + 7^{n+1}}$.

Solution. $a_n = \frac{2^{2n} + (-3)^{n-1}}{5^{n+2} + 7^{n+1}} = \frac{4^n - \frac{1}{3} \cdot (-3)^n}{25 \cdot 5^n + 7 \cdot 7^n} = \left(\frac{4}{7}\right)^n \cdot \frac{1 - \frac{1}{3} \cdot \left(-\frac{3}{4}\right)^n}{25 \cdot \left(\frac{5}{7}\right)^n + 7} \longrightarrow 0 \cdot \frac{1 - 0}{0 + 7} = 0.$

Here we used that $a^n \longrightarrow 0$ if $|a| < 1$.