There are no classes which contain objects of more than one type. Accordingly there is a universal class and a null-class proper to each type of object. But these symbols need not be distinguished, since it will be found that there is no possibility of confusion. Similar remarks apply to relations.

Descriptions. By a "description" we mean a phrase of the form "*the* so-and-so" or of some equivalent form. For the present, we confine our attention to *the* in the singular. We shall use this word strictly, so as to imply uniqueness; *e.g.* we should not say "A is *the* son of B" if B had other sons besides A. Thus a description of the form "the so-and-so" will only have an application in the event of there being one so-and-so and no more. Hence a description requires some propositional function $\phi\hat{x}$ which is satisfied by one value of x and by no other values; then "the x which satisfies $\phi\hat{x}$" is a description which definitely describes a certain object, though we may not know what object it describes. For example, if y is a man, "x is the father of y" must be true for one, and only one, value of x. Hence "the father of y" is a description of a certain man, though we may not know *what* man it describes. A phrase containing "the" always presupposes some initial propositional function not containing "the"; thus instead of "x is the father of y" we ought to take as our initial function "x begot y"; then "the father of y" means the one value of x which satisfies this propositional function.

If $\phi\hat{x}$ is a propositional function, the symbol "$(\iota x)(\phi x)$" is used in our symbolism in such a way that it can always be read as "the x which satisfies $\phi\hat{x}$." But we do not define "$(\iota x)(\phi x)$" as standing for "the x which satisfies $\phi\hat{x}$," thus treating this last phrase as embodying a primitive idea. Every use of "$(\iota x)(\phi x)$," where it apparently occurs as a constituent of a proposition in the place of an object, is defined in terms of the primitive ideas already on hand. An example of this definition in use is given by the proposition "$E!(\iota x)(\phi x)$" which is considered immediately. The whole subject is treated more fully in Chapter III.

The symbol should be compared and contrasted with "$\hat{x}(\phi x)$" which in use can always be read as "the x's which satisfy $\phi\hat{x}$." Both symbols are incomplete symbols defined only in use, and as such are discussed in Chapter III. The symbol "$\hat{x}(\phi x)$" always has an application, namely to the class determined by ϕx; but "$(\iota x)(\phi x)$" only has an application when $\phi\hat{x}$ is only satisfied by one value of x, neither more nor less. It should also be observed that the meaning given to the symbol by the definition, given immediately below, of $E!(\iota x)(\phi x)$ does not presuppose that we know the meaning of "one." This is also characteristic of the definition of any other use of $(\iota x)(\phi x)$.

We now proceed to define "$E!(\iota x)(\phi x)$" so that it can be read "the x satisfying ϕx exists." (It will be observed that this is a different meaning of existence from that which we express by "$\exists$.") Its definition is

$$E!(\iota x)(\phi x) . = : (\exists c) : \phi x . \equiv_x . x = c \quad \text{Df},$$

i.e. "the x satisfying $\phi\hat{x}$ exists" is to mean "there is an object c such that ϕx is true when x is c but not otherwise."

The following are equivalent forms:

$$\vdash :. E!(\imath x)(\phi x) . \equiv : (\exists c) : \phi c : \phi x . \supset_x . x = c,$$

$$\vdash :. E!(\imath x)(\phi x) . \equiv : (\exists c) . \phi c : \phi x . \phi y . \supset_{x,y} . x = y,$$

$$\vdash :. E!(\imath x)(\phi x) . \equiv : (\exists c) : \phi c : x \neq c . \supset_x . \sim\phi x.$$

The last of these states that "the x satisfying $\phi\hat{x}$ exists" is equivalent to "there is an object c satisfying $\phi\hat{x}$, and every object other than c does not satisfy $\phi\hat{x}$."

The kind of existence just defined covers a great many cases. Thus for example "the most perfect Being exists" will mean:

$$(\exists c) : x \text{ is most perfect} . \equiv_x . x = c,$$

which, taking the last of the above equivalences, is equivalent to

$$(\exists c) : c \text{ is most perfect} : x \neq c . \supset_x . x \text{ is not most perfect.}$$

A proposition such as "Apollo exists" is really of the same logical form, although it does not explicitly contain the word *the*. For "Apollo" means really "the object having such-and-such properties," say "the object having the properties enumerated in the Classical Dictionary*." If these properties make up the propositional function ϕx, then "Apollo" means "$(\imath x)(\phi x)$," and "Apollo exists" means "$E!(\imath x)(\phi x)$." To take another illustration, "the author of Waverley" means "the man who (or rather, the object which) wrote Waverley." Thus "Scott is the author of Waverley" is

$$\text{Scott} = (\imath x)(x \text{ wrote Waverley}).$$

Here (as we observed before) the importance of *identity* in connection with descriptions plainly appears.

The notation "$(\imath x)(\phi x)$," which is long and inconvenient, is seldom used, being chiefly required to lead up to another notation, namely "$R\text{‘}y$," meaning "the object having the relation R to y." That is, we put

$$R\text{‘}y = (\imath x)(xRy) \quad \text{Df.}$$

The inverted comma may be read "of." Thus "$R\text{‘}y$" is read "the R of y." Thus if R is the relation of father to son, "$R\text{‘}y$" means "the father of y"; if R is the relation of son to father, "$R\text{‘}y$" means "the son of y," which will only "exist" if y has one son and no more. $R\text{‘}y$ is a function of y, but not a propositional function; we shall call it a *descriptive* function. All the ordinary functions of mathematics are of this kind, as will appear more fully in the sequel. Thus in our notation, "sin y" would be written "sin ‘y," and "sin" would stand for the relation which sin ‘y has to y. Instead of a variable descriptive function fy, we put $R\text{‘}y$, where the variable relation R takes the

* The same principle applies to many uses of the proper names of existent objects, *e.g.* to all uses of proper names for objects known to the speaker only by report, and not by personal acquaintance.

place of the variable function f. A descriptive function will in general exist while y belongs to a certain domain, but not outside that domain; thus if we are dealing with positive rationals, $\sqrt{y}$ will be significant if y is a perfect square, but not otherwise; if we are dealing with real numbers, and agree that "$\sqrt{y}$" is to mean the *positive* square root (or, is to mean the negative square root), $\sqrt{y}$ will be significant provided y is positive, but not otherwise; and so on. Thus every descriptive function has what we may call a "domain of definition" or a "domain of existence," which may be thus defined: If the function in question is $R\text{‘}y$, its domain of definition or of existence will be the class of those arguments y for which we have $\text{E}!\, R\text{‘}y$, *i.e.* for which $\text{E}!\,(\imath x)(xRy)$, *i.e.* for which there is one x, and no more, having the relation R to y.

If R is any relation, we will speak of $R\text{‘}y$ as the "associated descriptive function." A great many of the constant relations which we shall have occasion to introduce are only or chiefly important on account of their associated descriptive functions. In such cases, it is easier (though less correct) to begin by assigning the meaning of the descriptive function, and to deduce the meaning of the relation from that of the descriptive function. This will be done in the following explanations of notation.

Various descriptive functions of relations. If R is any relation, the *converse* of R is the relation which holds between y and x whenever R holds between x and y. Thus *greater* is the converse of *less*, *before* of *after*, *cause* of *effect husband* of *wife*, etc. The converse of R is written* $\text{Cnv}\text{‘}R$ or $\breve{R}$. The definition is

$$\breve{R} = \hat{x}\hat{y}\,(yRx) \quad \text{Df},$$

$$\text{Cnv}\text{‘}R = \breve{R} \quad \text{Df}.$$

The second of these is not a formally correct definition, since we ought to define "Cnv" and deduce the meaning of $\text{Cnv}\text{‘}R$. But it is not worth while to adopt this plan in our present introductory account, which aims at simplicity rather than formal correctness.

A relation is called *symmetrical* if $R = \breve{R}$, *i.e.* if it holds between y and x whenever it holds between x and y (and therefore vice versa). Identity, diversity, agreement or disagreement in any respect, are symmetrical relations. A relation is called *asymmetrical* when it is incompatible with its converse, *i.e.* when $R \dot{\frown} \breve{R} = \dot{\Lambda}$, or, what is equivalent,

$$xRy \,.\, \supset_{x,y} .\, \sim (yRx).$$

Before and after, greater and less, ancestor and descendant, are asymmetrical, as are all other relations of the sort that lead to *series*. But there are many asymmetrical relations which do not lead to series, for instance, that of

* The second of these notations is taken from Schröder's *Algebra und Logik der Relative*.