AN ARGUMENT AGAINST THE PLAUSIBILITY OF CHURCH'S THESIS

LÁSZLÓ KALMÁR
Bolyai Institute, The University, Szeged, Hungary

1. In his famous investigations on unsolvable arithmetical problems [1], Church used a working hypothesis, viz. the identification of the notion of effectively calculable functions with that of general recursive (or, equivalently, λ-definable) functions. This working hypothesis is known under the name Church's thesis. It has several equivalent forms – e.g. Turing's identification of the notion of effectively calculable functions with that of functions computable by means of a Turing machine [10], or Markov's principle of the normalizability of algorithms [6] or [7] – which are generally accepted in the investigations on unsolvable mathematical problems.

In the present contribution, I shall not disprove Church's thesis. Church's thesis is not a mathematical theorem which can be proved or disproved in the exact mathematical sense, for it states the identity of two notions only one of which is mathematically defined while the other is used by mathematicians without exact definition. Of course, Church's thesis can be masked under a definition: we call an arithmetical function effectively calculable if and only if it is general recursive, venturing however that once in the future, somebody will define a function which is on the one hand, not effectively calculable in the sense defined thus, on the other hand, its value obviously can be effectively calculated for any given arguments. Similarly, in defining a problem, containing a parameter which runs through the natural numbers, to be solvable if and only if its characteristic function [1]) is general recursive, one takes the risk that somebody in the future will solve a problem

[1]) The characterestic function of a problem with parameter is defined as the function taking the value 1 or 0 for a value of the parameter according as "yes" of "no" is the correct answer to the particular case of the problem corresponding to this value of the parameter.

which is unsolvable under this definition. For this reason, it seems me better to regard such statements as Church's thesis, or the identity of solvable problems with those having a general recursive characteristic function as propositions rather than definitions [1]), however, not mathematical but "pre-mathematical" ones. The more than two pages [2]) of CHURCH's paper [1] filled with plausibility (hence pre-mathematical) arguments for his thesis, show that his opinion about this question does not differ much from mine.

Now, I shall give some arguments, of course likewise pre-mathematical ones, *against* Church's thesis. In these arguments, I shall freely use the *tertium non datur*, hence, they do not claim to be accepted by adherents of contructivistic doctrines which reject the *tertium non datur*.

I do not discuss the question, dealt with by R. Péter [3]), whether every general recursive function can be regarded as effectively calculable. However, I disbelieve in the converse statement, viz. that every effectively calculable function is general recursive, provided we regard as effectively calculable any arithmetical function, the value of which can be effectively calculated for any given arguments in a finite number of steps, irrespective how these steps are and how they depend on the arguments for which the function value is to be calculated [4]). In particular, I do not suppose the calculation method to be "uniform". By the way, such an uniformity supposition seems to have no objective meaning. For a school-boy, the method for the solution of the diverse arithmetical problems he has to solve does not seem uniform until he learns to solve equations; and several methods in algebra, geometry and theory of numbers which are now regarded group-theoretic methods were not considered as uniform before group-theory has been discovered.

[1]) For a similar opinion, see [9], footnote 8 on p. 105. However, I do not agree in considering the really fundamental discovery of Church as a discovery in the (absolute) limitations of the mathematizing power of Homo Sapiens.

[2]) pp. 356–358.

[3]) In a lecture on the same conference [8].

[4]) Of course, the definition of the calculation steps is not allowed to require a previous knowledge of the function value to be calculated but it has to be based only on the definition of the function itself. Without this restriction, every arithmetical function would be "effectively calculable."

2. In order to confirm my disbelief in the general recursivity of every effectively calculable function, I shall not give an instance for an arithmetical function which is at the one hand, not general recursive, for which, on the other hand, an obvious method can be given for the effective calculation of its value in a finite number of steps for any given arguments. However, for Kleene's instance [1]) of a not general recursive function defined by

$$\psi(x)=\mu_y(\varphi(x, y)=0)=\begin{cases}\text{the least natural number } y \text{ for which} \\ \quad \varphi(x, y)=0 \text{ if there is such an } y, \\ 0 \quad \text{if there is no natural number } y \text{ such} \\ \quad \text{that } \varphi(x, y)=0\end{cases}$$

with an appropriate general recursive function φ of two arguments, I shall show that the supposition that ψ is not effectively calculable — as a matter of fact, a corollary of Church's thesis — has strange consequences.

Indeed, on the one hand, for any natural number p for which a natural number y with $\varphi(p, y)=0$ exists, an obvious method for the calculation of the least such y, i.e. of $\psi(p)$ can be given: calculate in succession the values $\varphi(p, 0)$, $\varphi(p, 1)$, $\varphi(p, 2)$, ... (each of which can be calculated, on account of the general recursivity of φ, in a finite number of steps), until we obtain a natural number q for which we have $\varphi(p, q)=0$ and take this q. On the other hand, for any natural number p for which we can prove, not in the frame of some fixed postulate system but by means of arbitrary — of course, correct — arguments that no natural number y with $\varphi(p, y)=0$ exists, we have also a method to calculate the value $\psi(p)$ in a finite number of steps: prove that no natural number y with $\varphi(p, y)=0$ exists, which requires in any case but a finite number of steps, and gives immediately the value $\psi(p)=0$. Hence, supposing that ψ is not effectively calculable and applying the *tertium non datur* — which has been utilized already in the definition of the function ψ — we infer the existence of a natural number p for which, on the one hand, *there is no natural number y such that* $\varphi(p, y)=0$, on the other hand, *this fact cannot be proved by any correct means* — a consequence of Church's thesis which seems very unplausible.

[1]) See [5], p. 741, Theorem XIV.

3. The proposition stating that, for this p, there is a natural number y such that $\varphi(p, y)=0$, would be undecidable, with other words, the problem if this proposition holds or not, would be unsolvable, not in Gödel's sense of a proposition neither provable nor disprovable in the frame of a fixed postulate system, nor in Church's sense of a problem with a parameter for which no general recursive method exists to decide, for any given value of the parameter in a finite number of steps, which is the correct answer to the corresponding particular case of the problem, "yes" or "no". As a matter of fact, the problem, if the proposition in question holds or not, does not contain any parameter and, supposing Church's thesis, *the proposition itself can be neither proved nor disproved*, not only in the frame of a fixed postulate system, but *even admitting any correct means*. It cannot be proved for it is false and it cannot be disproved for its negation cannot be proved. According to my knowledge, this consequence of Church's thesis, viz. the existence of a proposition (without parameter) which is undecidable in this, *really absolute* sense, has not been remarked so far.

However, this "absolutely undecidable proposition" has a defect of beauty: we can decide it, for we know, it is false. Hence, *Church's thesis implies the existence of an absolutely undecidable proposition which can be decided viz., it is false, or*, in another formulation, *the existence of an absolutely unsolvable problem with a known definite solution*, a very strange consequence indeed.

4. Of course, this consequence of Church's thesis cannot be proved by any correct means, for its proof had to contain, on the one hand, a disproof of the proposition in question, on the other hand, a proof that it cannot be disproved (neither proved), which is impossible.

Taking into account the form of the proposition in question, viz. $\exists y \mathbf{P}(y)$ with a general recursive, hence effectively decidable property $\mathbf{P}$, we can show that even its absolute undecidability cannot be proved by any correct means. Indeed, if a proposition $\exists y \mathbf{P}(y)$ with an effectively decidable $\mathbf{P}$ is true, i.e. a natural number q with the property $\mathbf{P}$ exists, then this q can be found and the question if $\mathbf{P}(q)$ holds or not can be decided in a finite number of steps, of course, the latter with the result that $\mathbf{P}(q)$ holds. Hence, the question whether $\exists y \mathbf{P}(y)$ holds can be decided, viz. with the result "yes", for $\exists y \mathbf{P}(y)$ is a consequence of $\mathbf{P}(q)$. By contraposition, if a proposition of the form $\exists y \mathbf{P}(y)$ with an effectively decidable $\mathbf{P}$ is undecidable, then it does not hold. Hence, if the undecibility of such a proposition $\exists y \mathbf{P}(y)$ could be proved, then the negation of that proposition could be proved too, thus, the proposition in question could be decided, hence it would not be really undecidable, which is impossible if only correct means are allowed.

The fact that some consequences of Church's thesis cannot be proved by any correct means can be regarded, I think, as arguments against its plausibility.

5. In another paper [2], Church has given a definition of the notion of constructivity of ordinals belonging to the second number class. In this paper, he utters as follows: "It is my present belief that the definition is absolute ... — towards those who do not find this convincing the definition may perhaps be allowed to stand as a challenge, to find either a less inclusive definition which cannot be shown to exclude some ordinal which ought reasonably to be allowed as constructive, or a more inclusive definition which cannot be shown to include some ordinal of the second number class which cannot be seen to be constructive." I agree with Church that, supposing Church's thesis, his definition is all right. Hence, his above words amount to declare his thesis as a challenge, to find, instead of the class of general recursive functions, either a less inclusive class which cannot be shown to exclude some function which ought reasonably to be allowed as effectively calculable, or a more inclusive class which cannot be shown to include some arithmetical function which cannot be seen to be effectively calculable. Now, I assert as an answer to this form of Church's challenge: by subjoining all arithmetical functions ψ defined by an equation of the form $\psi(x) = \mu_y(\varphi(x, y) = 0)$ with a general recursive function φ of two arguments to the class of the general recursive functions, we obtain a more inclusive class satisfying the above requirements; moreover, for any such function ψ and any given natural number p, the following method allows to calculate the value $\psi(p)$ in a finite number of steps. Calculate in succession the values $\varphi(p, 0)$, $\varphi(p, 1)$, $\varphi(p, 2)$, ... and simultaneously try to prove by all correct means that none of them equals 0, until we find either a (least) natural number q for which $\varphi(p, q) = 0$ or a proof of the proposition stating that no natural number y with $\varphi(p, y) = 0$ exists; and consider in the first case this q, in the second case 0 as result of the calculation. I cannot prove my assertion, just as Church did not prove his thesis. However, if a challenge might be allowed as an argument, my above assertion may perhaps be allowed to stand as a challenge, to find a general recursive function φ and a natural number p and to prove that for this φ and p, the above method fails. I am in the agreable position of not to have to be afraid that somebody will do this. For he ought to prove, for his φ and p, amoung others that calculating in succession the values $\varphi(p, 0)$, $\varphi(p, 1)$, $\varphi(p, 2)$, ..., we never get 0, i.e., there is no y for which $\varphi(p, y) = 0$; if he can do this, than he cannot prove (by correct means) this proposition to be not provable.

6. In my arguments I used the somewhat vague concept of a proof by arbitrary correct means [1]); perhaps, this may be allowed in a pre-mathematical argumentation. However, in the argument of section 2, I can replace this concept by a more definite one. Indeed, let be S a system of equations serving as a general recursive definition of the function φ; let be $\varphi_1, \varphi_2, \ldots, \varphi_r$ $(= \varphi)$ the function symbols occurring in S. Consider the postulate system P defined as follows [2]). The formulae of P are formed out of the constant symbol 0 as well as numerical variables by means of the function symbols $\varphi_1, \varphi_2, \ldots, \varphi_r$ (with prescribed numbers of arguments), the equality symbol, further by means of predicate variables, truth-

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[1]) At the Third All-Union Mathematical Congress of the Soviet Union, I got informed that Novikov proposed the hypothesis, known under the name Novikov's prognosis, that this concept can be defined exactly, in the same sense as that of effective calculability by its identification with general recursivity. As to my opinion about this hypothesis, see section 7 of the present paper.

[2]) In [5], loc. cit., φ has been proved to be primitive recursive. By refining the same arguments, φ can be replaced by an *elementary* function. Hence P can be replaced by the postulate system P' defined as follows. The formulae of P' are formed out of the symbol 0 as well as numerical variables by means of the function and operator symbols $\ldots'$, $(\ldots + \ldots)$, $|\ldots - \ldots|$, $(\ldots \cdot \ldots)$, $[\ldots / \ldots]$, $\sum_{\ldots \leqq \ldots} \ldots$, $\prod_{\ldots \leqq \ldots} \ldots$ (for ... before $\leqq$ always a numerical variable has to stand), the symbols for variable descriptive

functions and quantifiers. The postulates of P are the equations belonging to S as well as those of the lower predicate calculus with identity; also, the rules of inference of P are those of the lower predicate calculus. Moreover, consider the consistent extensions of P by means of some formulae of P as new postulates. Now, the arguments of section 2 can be repeated with the concept of a proof within such a consistent extension of P instead of that of a proof by arbitrary correct means. Indeed, if in such an extension of P, we can prove for a natural number p the formula which formalizes the proposition stating the non-existence of a natural number y with $\varphi(p, y)=0$, then this proposition is true indeed, hence we have $\psi(p)=0$. For in the opposite case we should have $\varphi(p, q)=0$ for some natural number q; hence by the definition of the general recursive functions, the formula of P formalizing this fact, therefore by the rules of the lower predicate calculus, the formula of P which formalizes the proposition stating the existence of a natural number y with $\varphi(p, y)=0$ could be proved in the same extension of P as well, contrarily to its consistency. Hence, the same argument as in section 2 shows that Church's thesis implies the existence of a natural number p for which, on the one hand, there is no natural number y such that $\varphi(p, y)=0$, on the other hand, the formula of P formalizing this fact cannot be proved in any consistent extension of P.

However, this consequence of Church's thesis can be easily disproved. Indeed, subjoining the formula of P which formalizes

functions and predicates, the equality symbol, the truth functions and quantifiers. The postulates of P' are the formulae

$$
\begin{array}{ll}
(a+0)=a, & (a\cdot 0)=0, \\
(a+b')=(a+b)', & (a\cdot b')=((a\cdot b)+a), \\
|(a+b)-a|=b, & [a/0]=0, \\
|a-(a+b)|=b, & [(((a+c')\cdot b)+a)/(a+c')]=b, \\
\sum_{x\leqq 0} f(x)=f(0) & \prod_{x\leqq 0} f(x)=f(0), \\
\sum_{x\leqq a'} f(x)=(\sum_{x\leqq a} f(x)+f(a')), & \prod_{x\leqq a'} f(x)=(\prod_{x\leqq a} f(x)\cdot f(a'))
\end{array}
$$

as well as those of the lower predicate calculus with identity; the rules of inference of P' are those of the lower predicate calculus with variables for descriptive functions. In the abstract of my lecture on the Conference, I used this postulate system; however, it does not give any advantage.

the proposition stating that $\varphi(p, a)\neq 0$ (with a free variable a), we should get a consistent extension of P. This follows e.g. from Gentzen's proof for the consistency of the postulate system of arithmetic [1]). Indeed, we should have $\varphi(p, q)\neq 0$ for any natural number q, i.e., the new postulate is a verifiable formula of P. On the other hand, in this extension of P, by means of the rules of the lower predicate calculus, we can prove the formula of P which formalizes the proposition stating that there is no natural number y for which $\varphi(p, y)=0$, contrarily to the above consequence of Church's thesis.

Formally, this argument seems like an indirect disproof of Church's thesis. However, in order to calculate the value $\psi(p)$ for a natural number p for which there is no natural number y with $\varphi(p, y)=0$, we have to give a consistent extension of P in the frame of which the formula of P formalizing this fact can be proved. Now, it is easy to define such an extension by means of the subjunction mentioned above; however, in order to prove its consistency, we have to prove (by some correct means) the verifiability of the new postulate, i.e. that there is no natural number y for which we have $\varphi(p, y)=0$. Hence, the calculation method, for any given p, of the value $\psi(p)$ refers still to the concept of an arbitrary correct proof. Therefore, the new form of my argument is as heuristic as the old one.

7. By the above arguments, I tried to show some motives of my opinion which can be summarized as follows. There are pre-mathematical concepts which must remain pre-mathematical ones, for they cannot permit any restriction imposed by an exact mathematical definition. Amoung these belong, I am convinced, such concepts as that of effective calculabity, or of solvability, or of provability by arbitrary correct means, the extension of which cannot cease to change during the development of Mathematics [2]).

[1]) See [3] or [4]. Of course, also a simpler proof is possible for the postulate system in question is but a rudiment of that of arithmetic.

[2]) Of course, from my above arguments other consequences can be drawn, if one wants to do so. For instance, one can insist upon Church's thesis and regard these arguments as quasi-refutations of the *tertium non datur*. So did Markov during the Third All-Union Mathematical Congress in Moscow 1956.

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Constructivity in Mathematics

COUNTABLE FUNCTIONALS

S. C. KLEENE

Department of Mathematics, University of Wisconsin

We call a functional "countable", if each value of it is determined by a finite amount of information about its function argument. For the items of information by which such a functional is given in its entirety are countable (and not of cardinality $2^{\aleph_0}$ as they are for an arbitrary functional of type 2, or $2^{2^{\aleph_0}}$ for type 3, etc.). Countable functionals have applications in discussing constructive interpretations of classical mathematics [1]).

We presuppose familiarity with some of the notation and terminology of [3] [2]), but not except in § 4 and 5.4–5.8 with our

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[1]) They are so applied by Kreisel [6] in studies of an (unpublished) interpretation of intuitionism by Gödel. Our own interpretation of intuitionism using functionals, which is an unpublished extension (and modification) of [1] along the lines proposed in [2], uses only recursive functionals of type 2, which are all of them countable.

The present investigation of countable functionals arose as an offshoot from the work reported in [5], in response to a question raised (in 1956) by Kreisel whether a certain property of recursive functions (cf. [3] Theorem III p. 279) extends to recursive functionals. It does not with the arbitrary functional arguments used in [5] ([5] LII), but does for recursively countable functionals (Theorem 3 below).

Meanwhile Kreisel formulated a "continuity" property of functionals with the same aim as our countability property. Subsequently we learned, at the Summer Institute for Mathematical Logic, sponsored by the American Mathematical Society and held at Cornell University, July 1–August 2, 1957, of work by Martin Davis in this direction.

Publishers' note. Summaries of lectures at the Summer Institute dealing with Kleene's interpretation of intuitionism, Davis' work, and constructive functionals, appear in the Appendix of this volume on pages 281, 285, 290, respectively.

[2]) Cf. [3] pp. 538 ff. Also we use the notations

$$\bar{\psi}(x) = \prod_{i<x} p_i^{\psi(i)+1},\ \bar{\psi}(x;y) = \prod_{i<x} p_i^{\psi(i,y)+1},\ \bar{\psi}(x,y) =$$

$$= \prod_{i<x} \exp \prod_{j<y} p_i^{\psi(i,j)+1}, \text{ etc.; } \mathrm{Seq}\,(w) \equiv w \neq 0\ \&\ (i)_{i<\mathrm{lh}(w)}[(w)_i \neq 0],$$

$$\mathrm{Ext}\,(w,u) \equiv \mathrm{Seq}\,(w)\ \&\ (Ex)_{x \leqq \mathrm{lh}(w)}[u = \prod_{i<x} p^{(w)_i}].$$