# Ergodic Hypothesis for Cylindric Billiards: The Erdőtarcsa Conjecture (N. Simányi and D. Szász)

**Definition:** *Cylindric billiards* are toric billiards where the scatterers are cylinders. Suppose that the bases of the cylinders are stricly convex, a property ensuring that the scatterers are convex, and the arising billiard is semi-dispersive.

The configuration space of a cylindric billiard is $\mathbf{Q} = \mathbb{T}^d \setminus (C_1 \cup \ldots \cup C_k)$, where $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ $(d \geq 2)$ is the unit torus. The cylindric scatterer $C_i$ $(i = 1, \ldots, k)$ is

$$C_i = \{a + l + t_i \mid a \in A_i,\ l \in B_i,\ \}/\mathbb{Z}^d.$$

where $A_i \subset \mathbb{R}^d$ is a lattice subspace, $L_i = A_i^{\perp}$ is its orthocomplement (we assume that $\dim L_i \geq 2$), $B_i \subset L_i$ is a strictly convex, compact domain with a $C^2$-smooth boundary (assume, for simplicity that $0 \in \mathrm{Int} B_i$) and finally $t_i \in \mathbb{R}^d$. The phase space $\mathbf{M}$ of our billiard will be the unit tangent bundle of $\mathbf{Q}$, i. e. $\mathbf{M} = \mathbf{Q} \times \mathbb{S}^{d-1}$. (Here, as usual, $\mathbb{S}^{d-1}$ is the $d-1$-dimensional unit sphere.) (In order to avoid further unnecessary complications, we also assume that the interior of the configuration space $\mathbf{Q} = \mathbb{T}^d \setminus (C_1 \cup \ldots \cup C_k)$ is connected.)

For any cylinder $C_i$ $(i = 1, \ldots, k)$ let us consider the group $\mathcal{G}_i$ consisting of all orthogonal transformations $U \in SO(d)$ that leave the points of the subspace $A_i$ fixed. Then we consider the subgroup $\mathcal{G}$ of $SO(d)$ algebraically generated by all such groups $\mathcal{G}_i : 1 \leq i \leq k$.

**The Erdőtarcsa Conjecture (Simányi-Szász, 2000)** *For every cylindric billiard flow $(\mathbf{M}, \{S^t\}, \mu)$ the transitivity of the action of $\mathcal{G}$ on the velocity sphere $\mathbb{S}^{d-1}$ is equivalent to the hyperbolic ergodicity (or, equivalently, to the hyperbolicity and Bernoulli property) of the billiard map.*

**Remark** Hard ball systems are particular cases of cylindric billiards and consequently the *Boltzmann-Sinai Ergodic Hypothesis* for hard ball systems is weaker that the Erdőtarcsa Conjecture. Indeed, for every pair of balls, the condition that their centers can not get closer than the sum of their radii defines a (lattice) cylinder in the above sense. Moreover, as shown in the aforementioned work of Simányi and Szász, the cylinders appearing as the scatterers of a hard ball system satisfy the transitivity condition. For the Erdőtarcsa Conjecture the strongest result is obtained by N. Simányi in a recent manuscript: Proving the Ergodic Hypothesis for Billiards with disjoint Cylindric Scatterers, preprint, July 2002.

**History** The Erdőtarcsa Conjecture was formulated in the work: N. Simányi-D. Szász: Non-integrability of Cylindric Billiards and Transitive Lie Group Actions, *Ergodic Theory and Dynamical Systems*, 20 (2000), 593-610. Its following, weaker version already appeared in D. Szász: The K-property of 'Orthogonal' Cylindric Billiards, *Commun. Math. Phys.* 160 (1994), 581-597.

**Conjecture (Szász, 1994)** For every cylindric billiard flow $(\mathbf{M}, \{S^t\}, \mu)$ the existence of a single sufficient (for the notion of sufficiency see Problem "Fundamental

Theorem") trajectory is equivalent to the hyperbolic ergodicity (or, equivalently, to the hyperbolicity and Bernoulli property) of the billiard map.