Fundamental Theorem for smooth billiards in high dimensions

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1 Problem

Prove the Fundamental Theorem of Semi-dispersing billiards for the case when scatterers are C_3 smooth.

2 Motivation

The Fundamental Theorem (FTH) is the key tool in proving *local ergodicity* of semi-dispersing billiards and also many focusing billiards.

3 Discussion

The precise formulation of the theorem is long, so we postpone it after the discussion. To understand most of the discussion, a rough understanding of the statement is enough:

Roughly speaking: The Fundamental Theorem states that through *many points* of the phase space, one can draw *long stable and unstable manifolds.*

3.1 History

The two-dimensional case had been settled in Sinai's celebrated work, [S(1970)]. Seventeen years had elapsed until the multidimensional generalization given by Chernov and Sinai appeared, [S-Ch(1987)]. It offered a quite involved, but in essence very transparent formulation of the theorem and a delicate proof. A self-contained exposition of the original ideas with detailed conditions and arguments were provided in [K-S-Sz(1990)] where a slightly more general, the so called "transversal" version of the fundamental theorem was announced — mainly with its application to three-billiards in mind. Several other papers have appeared in the '90s with nice expositions of the theorem, even for classes of dynamical systems more general than the original semi-dispersing billiard setting (eg. Hamiltonian systems with singularities in [L-W(1995)]).

3.2 Difficulty:

However, all of these papers assumed that for all powers of the dynamics the singularity set (the set of phase points where dynamics is not continuous) is a finite collection of one-codimensional smooth and compact submanifolds. This assumption was necessary to estimate the size of certain neighbourhoods of the singularity set on which unstable manifolds end. In [B-Ch-Sz-T(2000)] this was shown *not* to be satisfied in the high-dimensional case. In [B-Ch-Sz-T(2002)], a proof of the FTH was given with the smoothness assumption replaced by the so-called "Lipschitz decomposability" assumption. The assumption was shown to hold in the case when scatterers have *algebraic* boundaries. The case of more general scatterers remained open.

So the way to go seems to be to *either* check Lipscitz decomposability for the singularity set (of iterates of the dynamics) for high-dim. billiards with any C_3 smooth scatterers, *or* find some other means to control the size of neighbourhoods of the singularity set.

4 (More) Precise formulation

of the version of the FTH in [B-Ch-Sz-T(2002)]

4.1 Prerequisites

Definition 4.1. Singularities. *Consider the set of tangential reflections, i.e.*

$$\mathcal{R} := \{(q, v) \in \partial M \mid \langle v, n(q)\rangle = 0\}.$$

It is easy to see that the map T is not continuous at the set $T^{-1}\mathcal{R}$. As a consequence, the singularity set for a higher iterate T^n is

$$\mathcal{R}^{(n)} = \cup_{i=1}^{n} \mathcal{R}^{-i},$$

where in general $\mathcal{R}^k = T^k\mathcal{R}$.

Notation

For any $n \in \mathbb{N}$, Δ_n stands for the set of doubly singular phase points up to order n, i.e. $x \in \partial M$ belongs to Δ_n whenever there are indices $k_1 \neq k_2$, $|k_i| \leq n$ such that both $T^{k_1}x$ and $T^{k_2}x$ are elements of $\mathcal{R}$. We will need a notation for phase points with regular or with at most once singular trajectories, thus we consider the following set:

$$\partial M^* := \partial M \setminus \bigcup_{n=1}^{\infty} \Delta_n$$

Definition 4.2. *local orthogonal manifolds (or fronts) are defined in the flow phase space the following way.*

Let $x = (q, v) \in M \setminus \partial M$ and consider a C^2-smooth codimension 1 submanifold $\Sigma' \subset Q \setminus \partial Q$ such that $q \in \Sigma'$ and $v = v(q)$ is the normal vector to Σ' at q. Denote by Σ the normal section of the unit tangent bundle on Q restricted to Σ'. Σ is called a local orthogonal manifold or simply a front. A front is said to be (strictly) convex whenever its second fundamental form $B_\Sigma(y) \geq 0$ ($B_\Sigma(y) > 0$) for every $y \in \Sigma$.

Definition 4.3. *Let us consider a nonsingular finite trajectory segment for the flow: $S^{[a,b]}x$, where $a < 0 < b$ and a, b, 0 are not moments of collision.*
$\mathcal{N}_0(S^{[a,b]}x)$, the neutral subspace at time 0 for the segment $S^{[a,b]}x$ is defined as follows:

$$\begin{aligned}\mathcal{N}_0(S^{[a,b]}x) := \quad \{ \quad & w \in \mathbb{R}^d : \exists(\delta > 0) s.t. \forall \alpha \in (-\delta, \delta) \\ & v(S^a(q(x) + \alpha w, v(x))) = v(S^a x) \& \\ & v(S^b(q(x) + \alpha w, v(x))) = v(S^b x)\}.\end{aligned}$$

Observe that $v(x) \in \mathcal{N}_0(S^{[a,b]}x)$ is always true, the neutral subspace is at least 1 dimensional. Neutral subspaces at time moments different from 0 are defined by $\mathcal{N}_t(S^{[a,b]}x) := \mathcal{N}_0(S^{[a-t,b-t]}(S^t x))$, thus they are naturally isomorphic to the one at 0.

Definition 4.4. *The non-singular trajectory segment $S^{[a,b]}x$ is sufficient if for some (and in that case for any) $t \in [a, b]$: $\quad dim(\mathcal{N}_t(S^{[a,b]}x)) = 1$. A point $x \in M^0$ is said to be sufficient if its entire trajectory $S^{(-\infty,\infty)}x$ contains a finite sufficient segment. Singular points are treated by the help of trajectory branches (see [K-S-Sz(1990)]): a point $x \in M^1$ (this precisely means that the entire trajectory contains one singular reflection) is sufficient if both of its trajectory branches are sufficient.*

All these concepts have their natural counterparts for the billiard map phase space ∂M. For example, a smooth piece $\Sigma \in \partial M$ of the image of a local orthogonal manifold in M is referred to as a front as well.

To precisely formulate the property that we propose instead of smoothness of the singularities, we need the following two definitions:

Definition 4.5. *A function $f : D \subset \mathbb{R}^{m-1} \to \mathbb{R}$ will be called locally Lipschitz (with Lipschitz contant L), if for any $x \in D$ there exists a neighbourhood $U \subset \mathbb{R}^{m-1}$ of x such that the restricted function $f\Big|_{D \cap U}$ is Lipschitz (with Lipschitz constant L).*

In all applications D will be open. Notice that in this case, f typically cannot even be extended to $\bar{D}$ in a continuous way.

Definition 4.6. *$H \subset \mathbb{R}^m$ will be called a (one-codimensional) open locally Lipschitz graph (with Lipschitz constant L), if we can choose an appropriate Cartesian coordinate system so that H becomes the graph of a locally Lipschitz function:*

$$H = \{(x, f(x) \mid x \in D\}$$

with some $D \subset \mathbb{R}^{m-1}$ and $f : D \to \mathbb{R}$ locally Lipschitz (with constant L).

We will be mainly interested in the case when the domain D is an open set in $\mathbb{R}^{m-1}$, then

– H will be called an open locally Lipschitz graph (even though it is not an open set in $\mathbb{R}^m$),

– and we will denote by ∂H the boundary of H as of a surface: $\partial H = \bar{H} \setminus H$.

Now we are able to define the regularity property that should replace the smoothness of singularities. This property, called 'Lipschitz decomposability' will be defined for subsets of $\mathbb{R}^m$ here. For Lipschitz decomposability of subsets of a Riemannian manifold, see Remark 4.8.

Definition 4.7. *Consider $H \subset \mathbb{R}^m$, and $L \in \mathbb{R}$. H will be called 'Lipschitz decomposable' (one-codimensional) subset with constant L if it can be decomposed into a finite number of open locally Lipschitz graphs and a small remainder set in the following way: There exist H^* and $H_1, ..., H_K$ such that:*

- $H \subset \bigcup_{i=1}^{K} \bar{H}_i \bigcup H^*$,
- $H_i \cap H_j = \emptyset$ *for any* $i \neq j$,
- *every H_i is a one-codimensional open locally Lipschitz graph (with constant L),*
- $\mathcal{L}^m \left(\left((\bigcup_{i=1}^{K} \partial H_i) \cup H^* \right)^{[\delta]} \right) = o(\delta)$.

The set H^* is included in the decomposition for technical reasons: we want to allow for sets H having parts of strictly higher codimension. This occurs generically if H is an algebraic subvariety of $\mathbb{R}^n$. Nevertheless we would like to note that such higher codimensional parts are not present in the singularities of semi-dispersing billiards.

Remark 4.8. Lipschitz decomposability in Riemannian manifolds.

Throughout the paper – and in particular in conjecture 4.9 below – subsets of a compact Riemannian manifold $\mathcal{M}$ are considered. For $H \subset \mathcal{M}$ Lipschitz decomposition is understood in terms of coordinate charts.

To be more precise, let us fix some convention related to the atlas $\{U_t, \psi_t\}_{t=1}^T$ for $\mathcal{M}$ first. It is important that $\mathcal{M}$ is compact thus we may consider a finite atlas. We say that the atlas is bi-Lipschitz *if all charts $\psi_t : U_t \to \mathbb{R}^m$ are bi-Lipschitz maps, i.e. both ψ_t and $(\psi_t)^{-1}$ are Lipschitz with some constant $K > 1$. All atlases considered are assumed to be bi-Lipschitz with a fixed constant. This ensures that Euclidean distance on $\mathbb{R}^m$ is comparable to Riemannian metric on the manifold. Note that bi-Lipschitzness – with Lipschitz constant arbitrarily close to one – can always be obtained by choosing the coordinate patches sufficiently small.*

As to the problem of Lipschitz decomposition, we will say that $H \subset \mathcal{M}$ is Lipschitz decomposable whenever a finite bi-Lipschitz atlas can be chosen, such that for all charts $\psi_t(H \cap U_t)$ is Lipschitz decomposable as a subset of $\mathbb{R}^m$, in the sense of Definition 4.7. [1]

The precise property that we expect the singularities of semi-dispersing billiards to have is formulated in the form of a conjecture:

Conjecture 4.9. *For any semi-dispersing billiard with a finite horizon there exists an $L \in \mathbb{R}$ such that for any integer N the set $\bigcup_{|n|\le N} \mathcal{R}^n$ of singularities of order up to N is 'Lipschitz decomposable' with constant L.*

It is worth noting that by introducing "transparent walls" (cf. [S-Ch(1987)]) any semi-dispersing billiard can be reduced to one with a finite horizon.

The statement of this conjecture will appear word by word among the conditions of the fundamental theorem for semi-dispersing billiards.

4.2 Formulation of the theorem

Before its formulation its is important to point out the **conditions** under which the modified proof of the theorem works. We use the notations just introduced.

Condition 4.10. (Chenov-Sinai Ansatz, Condition 3.1 from [K-S-Sz(1990)]). *For almost every point $x \in \mathcal{R}$ we have $x \in \partial M^*$ and, moreover, the positive semi-trajectory of the point x is sufficient.*

Condition 4.11. *There exists an $L \in \mathbb{R}$ such that for every $N \in \mathbb{N}$ the singularity set $\bigcup\limits_{|n|\le N} \mathcal{R}^n$ is 'Lipschitz decomposable' with constant L (cf. Conjecture 4.9).*

To formulate the fundamental theorem, we introduce the notion of regular coverings. Note that $m = 2d - 2$ is the dimension of the (Poincaré) phase space ∂M. The next definition will not be absolutely precise, for we omit some technical details for the sake of easier understanding. For a precise formulation please see Definition 3.4 in [K-S-Sz(1990)].

Definition 4.12. *Let us assume that for a point $x \in \partial M^*$ and its neighborhood $U(x)$ a smooth foliation $U(x) = \cup_{\alpha \in B^{d-1}} \Gamma_\alpha$ is given. The foliae Γ_α are $d-1$-dimensional manifolds uniformly transversal to all possible local stable manifolds (B^{d-1} is the standard $d-1$-dimensional open ball).*

The parametrized family of finite coverings

$$\mathcal{G}^\delta = \{ G_i^\delta \mid i = 1, ..., I(\delta) \} \qquad 0 < \delta < \delta_0$$

is a family of regular coverings iff:

[1] The delicate question how sensitive this notion of Lipschitz-decomposition is to the choice of the atlas needs further investigation.

1. *each G_i^δ is an open parallelepiped of dimension $2d-2$;*

2. *the $d-1$-dimensional faces of G_i^δ are all cubes with edge-length δ, moreover, they may belong to two different categories: the s-faces are 'parallel with leaves of the stable foliation' while the Γ-faces are 'parallel' with the leaves of the foliation Γ;*[2]

3. *For any point, the maximal number of parallelepipeds covering it is 2^{2d-2};*

4. *if $G_i^\delta \cap G_j^\delta \neq \emptyset$, then*
$$\mu_1(G_i^\delta \cap G_j^\delta) \geq c_1 \delta^{2d-2}$$
with c_1 independent of δ.

Some further convention: Given any G_i^δ its *s-jacket*, $\partial^s(G_i^\delta)$ is the union of those $(2d-3)$-dimensional faces of G_i^δ which contain at least one s-face of it. The Γ-jacket, $\partial^\Gamma(G_i^\delta)$ is defined similarly. Clearly, $\partial(G_i^\delta) = \partial^s(G_i^\delta) \cup \partial^\Gamma(G_i^\delta)$. We say that a stable manifold $\gamma^s(y)$ *intersects G_i^δ correctly* if:
$$\partial(G_i^\delta \cap \gamma^s(y)) \subset \partial^\Gamma(G_i^\delta).$$

Theorem 4.13. (The Fundamental Theorem) *We assume that:*

- *conditions 4.10 and 4.11 are satisfied;*
- *a sufficient phase point $x \in \partial M^*$ is given;*
- *a smooth transversal foliation Γ in a neighborhood U_0 of x is fixed;*
- *a constant $0 < \epsilon_1 < 1$ is chosen.*

Then there is a sufficiently small neighborhood $U_{\epsilon_1}(x)$ such that for any $U(x) \subset U_{\epsilon_1}(x)$ and for any family of regular coverings, the covering $\mathcal{G}^\delta$ can be divided into two disjoint subsets, $\mathcal{G}_g^\delta$ and $\mathcal{G}_b^\delta$ (called 'good' and 'bad'), in such a way that:
(I) For any $G_i^\delta \in \mathcal{G}_g^\delta$ and any s-face E^s of it, the set:
$$\{y \in G_i^\delta \mid \rho(y, E^s) < \epsilon_1 \delta \text{ and } \gamma^s(y) \text{ intersects correctly}\}$$
has positive relative μ_1-measure in G_i^δ.
(II)
$$\mu_1\left(\bigcup_{G_i^\delta \in \mathcal{G}_b^\delta} G_i^\delta\right) = o(\delta).$$

[2]More precisely if we consider the center of each parallelepiped $w_i^\delta \in G_i^\delta$, the s- and Γ- faces are parallel with the tangent planes $\mathcal{T}_{w_i^\delta}\gamma^s(w_i^\delta)$ and $\mathcal{T}_{w_i^\delta}\Gamma(w_i^\delta)$, respectively.

References

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