Exponential decay of correlations in multi-dimensional dispersing billiards

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1 Problem

Prove that Correlations for Lipschitz observables decay exponentially in multi-dimensional dispersing billiards.

2 Motivation

Exponential decay of correlations is a key step towards the investigation of statistical properties, e.g. transport coefficients in a physical system. Dispersing billiards seem to be the easiest to handle among multi-dimensional systems. On the other hand, understanding correlation decay in these systems would open the way towards the multi-dimensional Lorentz process, and also the utmost important system of hard balls – which is a multi-dimensional semi-dispersing system.

3 Discussion

Definition 3.1. *Consider a phase space M with a dynamics T and a T-invariant probability measure μ. We say that the dynamical system (M, T, μ) has exponential decay of correlations (EDC) , if for every $f, g : M \to \mathbb{R}$ Hölder-continuous pair of functions there exist constants $C < \infty$ and $a > 0$ such that for every $n \in \mathbb{N}$*

$$\left| \int_M f(x) g(T^n x) d\mu(x) - \int_M f(x) d\mu(x) \int_M g(T^n x) d\mu(x) \right| \leq C e^{-an}.$$

3.1 History

After a long history of weaker results, the breakthrough in the investigation of correlation decay, leading to a proof of *exponential* decay for *planar* dispersing billiards was the method of L.S. Young, [3]. The essence of her paper is a fine analysis of hyperbolic behaviour, i.e. the growth of unstable manifolds. Besides, the differential aspects of dynamics properties like distortion and curvature bounds are needed. These are related to the inhomogeneities of the derivative of the dynamics. Roughly speaking, in addition to the first, even the second derivative is to be investigated.

In his work [2], based on the paper of Young, N. Chernov gave certain technical conditions concerning unstable manifolds, which ensure exponential decay of correlations. He applied his theorem to prove exponential mixing for all major types of planar dispersing billiards (and thus specifically for those corresponding to Lorentz processes).

The conditions of Chernov's theorem are conjectured to hold for multi-dimensional dispersing billiards also, so a promising way to deal with the problem is to try to check these technical conditions. A precise formulation of these conditions would be too long, so we only list them by keywords here. The reader may find every detail in [2].

The properties to check are:

1. **uniform hyperbolicity**
2. uniform **curvature bounds** for stable and unstable manifolds
3. **distortion bounds** claiming that expansion factors do not vary too fast along unstable manifolds
4. **absolute continuity** property of the holonomy map taking one unstable manifold into another, defined by sliding along stable manifolds
5. **growth properties** of unstable manifolds

3.2 Difficulty:

Properties 1, 2, 3 and 4 were checked by P. Bálint, N. Chernov, D. Szász and I. P. Tóth in [1]. However, in the same paper, a major difficulty concerning the proof of 5 was demonstrated. Namely:

Due to uniform hyperbolicity, unstable manifolds are expanded as time evolves. On the other hand, at every iteration of the dynamics, they are chopped into pieces by the singularities (sets of phase points where the dynamics is not continuous). The essence of Condition 5 is that **expansion prevails** against cutting, so unstable manifolds do grow on the long run, and even have a nice shape with large interior and small boundary.

To prove these properties, one needs to handle the set of points that are "too close" to a singularity. Namely, the measure of a neighbourhood of certain radius of the singularity set has to be estimated. Everything would work out fine if we had a uniform bound on the *curvature of singularities*.

However, in [1] it was shown that there is no such bound. Worse than that, even a single singularity (of, say, the second iterate of the dynamics) has no bounded curvature, the singularity set is not even differentiable in certain "pathological" points. This spoils the above mentioned measure-theoretic estimates, and thus the proof of condition 5.

All in all, the proof of EDC for multi-dimensional dispersing billiards probably requires a better understanding of the structure of singularity manifolds of higher iterates of the dynamics.

References

[1] P. Bálint, N. I. Chernov, D. Szász and I. P. Tóth, *Geometry of Multi-dimensional Dispersing Billiards*, to appear in Asterisque

[2] N. Chernov, *Decay of correlations and dispersing billiards*, J. Statist. Phys. **94** (1999), 513–556.

[3] L.S. Young, *Statistical properties of dynamical systems with some hyperbolicity*, Annals of Math. **147** (1998), 585–650.