Softenings of dispersing billiards

Planar case

Consider the following natural modification of the planar Lorentz process. (For a definition of the Lorentz process, see the document "Some basic convantions" on this webpage.) All scatterers are circular disks, nevertheless, the point particle may enter these disks. I.e. the hard core collision law is replaced by Hamiltonian dynamics: motion is governed by some potential that is axis-symmetric inside the scatterers and vanishes identically outside.

When talking about the soft version of the model, as usual (see the above document) we consider the dispersing billiard corresponding to the Lorentz process. Whether hyperbolicity and ergodicity persist depends on the softening performed. On the one hand, there is quite a wide class of potentials for which the soft billiard system was proved to be K-mixing in [1] (actually, the first results in this direction date back to Kubo et. al., [5], thus a certain class of these ergodic potentials are often referred to as Kubo-type). It is also proved that under finiteness of horizon and some regularity conditions – that seem reasonable, nevertheless could be checked only for some specific potentials – the rate of mixing is exponential (see [2]). On the other hand, as discussed for example in [3] and [4], in case the resulting potential is smooth (more precisely C^4, as a function on $\mathbb{R}^2$), stable periodic orbits tend to appear and thus the system cannot be ergodic. Actually, it is not just the potential that matters, mixing properties are sensitive to the geometry of the disk configuration. For some potentials, ergodicity (and exponential decay of correlations) was proved in case the minimum length of free flight is suitably bounded below, while stable periodic orbits were constructed for configurations with scatterers not sufficiently far away.

In view of the discussion above the following interesting open problems naturally arise.

1. Is there a smooth potential for which the soft billiard system is ergodic? If yes, how stable is ergodicity with respect to perturbations of the configuration or of the full energy level? If no, how much non-smoothness is needed for hyperbolicity, ergodicity or for exponential mixing? Can the potential be, for example, C^1?

2. Is there an ergodic softened dispersing billiard for which correlations decay slower than exponential (eg. with some polynomial rate)?

3. How stable are the mixing properties with respect to perturbations of the geometric configuration or of the full energy level? Once – via eg. exponential decay of correlations – the existence of transport coefficents like the diffusion constant is established, how do these depend on the perturbations described above?

Remark. One of the most effective ways of investigating chaotic behaviour in softened dispersing billiards is by means of the so-called *rotation function* that can be calculated from the potential by means of typically complicated integrals (on details see [1], [2] or [3]). The results in references [1,2,3] essentially apply to softened billiards with suitable rotation functions. Thus to address the problems listed above it would be desirable to find effective means of obtaining rotation functions for a wider class of potentials.

Multidimensional case

A natural construction for a multi-dimensional soft Lorentz process is to consider a periodic configuration of balls with spherically symmetric potentials within them. This model has not yet been investigated in a mathematically rigorous way at all. Thus there are many open problems, for example:

4. Find a spatial (or multi-dimensional) soft billiard for which a complete proof of hyperbolicity (or a complete proof of ergodicity, K-mixing, exponential decay of correlations) can be performed!

5. Do the natural analogons of the so far investigated planar systems behave similarly in high dimensions? For example, are – at least for certain configurations and full energies – potentials of a constant value or with $r^{-\alpha}$ ($\alpha > 0$ fixed) singularities ergodic (exponentially mixing) in higher dimensions as well?

Though even more difficult, it is worth mentioning one more related problem. Consider N particles moving on the unit 2-torus and interacting via a potential of finite range. With a suitable analysis of the isomorphism between hard ball systems and cylindric billiards (see [3] and Problem "The Erdőtarcsa Conjecture") on this web page) results on softened palnar billiards imply ergodicity (or the lack of ergodicity) for the case $N = 2$. The case $N \geq 3$ is highly non-trivial as besides the difficulty of softening semi-dispersing billiards one needs to understand the mechanism of multiple interactions. The following problem of C. Liverani and D. Szász appeared originially in [6].

6. Let $N = 3$. Is it possible to find a Kubo-type interaction such that the system is ergodic?

Remark. The issue of higher dimensional soft billiards is a very little discovered one, thus it is hard to guess what the main difficulties will be. Nevertheless, problems related to anisotropy will definitely play an important role, just like in other high dimensional billiards (see the related problems of the homepage on multidimensional dispersing systems and stadia).

References:

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