Complexity property of singularities in high dimensions

Setup:

Consider the simplest high-dimensional dispersing billiard setting: balls are cut out of a (say, 3-dimensional) torus, and a point particle is moving uniformly in the remaining part, reflecting elastically from the balls (the 'scatterers'). Use the Poincaré phase space and the Poincaré map as defined in the document 'Some basic conventions'. The boundary of the Poincaré phase space is the set of 'tangential reflections', that is, the set of collision phase points in which the only touches the scatterer, but does not change the direction of its velocity Pre-images by (some iterate of) the dynamics T of this set are called *singularities*, because in these points (some iterate of) T is non-continuous.

Denote by Γ_n the set of n-step singularities, that is, phase points which will have a tangential reflection within n steps. Γ_n consists of many components. Let $K(n)$ be the *maximum* number of components of $Gamma_n$ that meet in a single phase point. This is the maximal **multiplicity** a singular phase point can have. $K(n)$ measures the complexity of the n-step singularity set Γ_n.

Problem:

Describe the asymptotic behaviour of $K(n)$ as $n \to \infty$. Prove at least that $K(n)$ grows subexponentially, that is, $K(n)/\lambda^n \to 0$ for any $\lambda > 1$.

Discussion:

the meaning of the word *component* depends on the context. Usually, points with the same symbolic collision sequence are ment – this means that they live the same history, always collide on the same scatterer. These were often called 'smooth components' untill it turned out that they are *not* smooth, especially near multiple collision points (see [1]). When algebraic billiards are discussed, *component* can mean algebraic components, which can be much more in number. Moreover, when we need to estimate the size of a neighbourhood of the singularity set, non-smoothness may impose a need to further cut the singularities into so-called 'Lipschitz components' introduced in [2].

Motivation:

In the study of correlation decay, it is important to estimate the size of neighbourhoods of Γ_n, especially locally. In these local estimates, the factor $K(n)$ comes up. Without having an upper bound on $K(n)$, the effect of singularities can not be treated.

Difficulty:

If scatterers were semi-transparent, such that part of the incomming particles would be reflected when entering the scatterer, but a part would enter straigth and only be reflected when trying to leave the scatterer, $K(n)$ would clearly grow exponentially. In fact, with every iteration of the dynamics, singualrity manifolds would typically be cut into two, so we would get $K(n) = C2^{n-1}$. We can only hope that the non-transparent scatterers 'cover' most of these components, not allowing the particle to take most of the pahts. In $d = 2$ this is easy to see. However, in higher dimensions, configurations can be constructed where every possible sequence does appear for quite large n-s. The problem is to understand how scatterers 'cover' each-other in high dimensions.

References:

[1] P Bálint, N. Chernov, D. Szász and I. P. Tóth; Geometry of Multidimensional Dispersing Billiards (to appear in 'Astrisque')

[2] P Bálint, N. Chernov, D. Szász and I. P. Tóth; Multidimensional Semi-Dispersing Billiards: Singularities and the Fundamental Theorem; Annales Henri Poincar, 3. 451-482 (2002)