Correlation decay in stadium-like billiards

Planar case

We will call a planar billiard stadium-like in case at least one of its boundary components is convex, however, it is not integrable. A prototype of such billiards is the Bunimovich stadium: two semicircles joint by two parallel segments in a C^1-smooth way. A significant class of such systems was proved to be hyperbolic, ergodic and K-mixing. Nevertheless, hyperbolicity is far from being uniform. This underlines the significance of investigating the rate of mixing for the billiard map T, which is expected to be polynomial (on related computer simulations see [6]). The only mathematically rigorous result obtained so far in this direction is due to Roberto Markarian (see [1]):

T-time correlations for Hölder continuous observables decay with rate $\frac{1}{n}$ in the Bunimovich stadium. In case the two semi-circles have different radii – and consequently they are joint by non-parallel segments – the decay rate is $\frac{1}{n^2}$.

It is a natural question what can be said about the mixing rates in other ergodic stadium-like billiards. Nevertheless, analogous multi-dimensional problems are even more interesting.

Multi-dimensional case

In a series of papers ([2-4]) L. Bunimovich and J. Reháček introduced and investigated multi-dimensional versions of stadium-like billiards. They considered rectangular regions in $\mathbb{R}^d$ ($d \geq 3$) to which finitely many so-called spherical caps were added. A spherical cap is created if one takes the union of the rectangular region and a ball arround some interior point of the region that intersects only one side. The resulting billiard domain Q is said to be a *Bunimovich-Reháček stadium* in case (cf. [3])

- the angles of all spherical caps are smaller than 90^o (this condition is, however, ensured by having all spheres arround interior points of the rectangular region),
- almost every trajectory enters at least once a spherical cap,
- each sphere (corresponding to some cap) is enclosed in a subregion of Q which has walls perpendicular to the one to which the cap is added.

The only results obtained so far on chaotic behavior in multi-dimensional stadium-like billiards are the hyperbolicity and the ergodicity of Bunimovich-Reháček stadia. Investigation of other billiard domains with similar properties seems to be highly nontrivial as indicated by remarkable counter-examples constructed by M. Wojtkowski (see [5]). Nevertheless, an even more interesting challenge would be to investigate

Open problem. How do correlations decay for the billiard map in Bunimovich-Reháček stadia?

Remark. The main difficulty in higher dimensional stadia is the strong anisotropy of the dynamics which is often referred to as *astigmatism.* It is worth mentioning that similar anisotropic behaviour appears in multi-dimensional dispersing billiards which compifies specifically the description and treatment of singularity manifolds. (See the related problems of this page, on decay of correlations for multi-dimensional dispersing billiards and on the complexity property for singularities.) Nevertheless, the mathematical theory of higher dimensional stadia is far less developed than that of multi-dimensional dispersing billiards.

References:

[1] R. Markarian; *Billiards with polynomial decay of correlations*, preprint

[2] L. Bunimovich and J. Reháček; *Nowhere dispersing* $3D$ *billiards with non-vanishing Lyapunov exponents*, Comm. Math. Phys **189** (1997), 729–757

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[4] L. Bunimovich and J. Reháček; *On the ergodicity of many-dimensional focusing billiards. Classical and quantum chaos.* Ann. Inst. H. Poincaré Phys. Théor. **68** (1998) 421–448

[5] M. Wojtkowski; *Linearly Stable Orbits in 3 Dimensional Billiards*, Comm. Math. Phys. **129** (1990) 319–327

[6] F. Vivaldi, G. Casati and I. Guanieri; *Origin of long-time tails in strongly chaotic systems*, Phys. Rev. Letters **51** (1983), 727–730.