A GEOMETRIC ESTIMATE ON THE NORM OF PRODUCT OF FUNCTIONALS

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Abstract. The open problem of determining the exact value of the n-th linear polarization constant c_n of $\mathbb{R}^n$ has received considerable attention over the last few years. This paper makes a contribution to the subject by providing a new lower bound on the value of $\sup_{\|\mathbf{y}\|=1} |\langle \mathbf{x}_1, \mathbf{y}\rangle \cdots \langle \mathbf{x}_n, \mathbf{y}\rangle|$, where $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are unit vectors in $\mathbb{R}^n$. The new estimate is given in terms of the eigenvalues of the Gram matrix $[\langle \mathbf{x}_i, \mathbf{x}_j\rangle]$ and improves upon earlier estimates of this kind. However, the intriguing conjecture $c_n = n^{n/2}$ remains open.

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1. Introduction

The present work contributes to study of the n-th linear polarization constant $c_n(\mathbb{R}^n)$ of the n-dimensional real Euclidean space. We begin with introducing some (more general) standard terminology and giving a short account of some related results.

Let X denote a Banach space over the real or complex field $\mathbb{K}$. A function $P: X \to \mathbb{K}$ is a continuous *n-homogeneous polynomial* if there exists a symmetric, continuous n-linear form $L: X^n \to \mathbb{K}$ such that $P(\mathbf{x}) = L(\mathbf{x}, \ldots, \mathbf{x})$ for all $\mathbf{x} \in X$. We define

$$\|P\| := \sup\{|P(\mathbf{x})| : \mathbf{x} \in B\}$$

where B denotes the unit ball of X. Considerable attention has been devoted to polynomials of the form $P(\mathbf{x}) = f_1(\mathbf{x}) f_2(\mathbf{x}) \ldots f_n(\mathbf{x})$, where $f_1, f_2, \ldots, f_n$ are bounded linear functionals on X. For any *complex* Banach space X Benítez, Sarantopoulos and Tonge [5] have obtained

$$\|f_1\| \, \|f_2\| \cdots \|f_n\| \leq n^n \|f_1 f_2 \ldots f_n\|,$$

and they also showed that, in general, the constant n^n is best possible. For *real* Banach spaces, Ball's solution [2] of the famous plank problem of Tarski gives the same result. For specific spaces, however, the general constant n^n can be lowered. This fact motivated the following

Definition 1.1. *(Benítez, Sarantopoulos, Tonge [5]) The n-th linear polarization constant of a Banach space X is defined by*

$$\begin{aligned} c_n(X) &:= \inf\{M : \|f_1\|\cdots\|f_n\| \le M\|f_1\cdots f_n\| \ (\forall f_1,\dots,f_n \in X^*)\} \\ &= 1/\inf_{f_1,\dots,f_n\in S_{X^*}} \sup_{\|\mathbf{x}\|=1} |f_1(\mathbf{x})\cdots f_n(\mathbf{x})|. \end{aligned}$$

The linear polarization constant of X is defined by

$$c(X) := \lim_{n\to\infty} c_n(X)^{\frac{1}{n}} \quad . \tag{1}$$

Let us recall that the above definition of $c(X)$ is justified since Révész and Sarantopoulos [11] showed that the limit (1) does exist. Moreover, they also showed (both in the real and complex cases) that $c(X) = \infty$ if and only if $\dim X = \infty$.

Note that it is easy to see that for any Banach space X we have

$$c_n(X) = \sup\{c_n(Y) : Y \text{is a subspace of} X,\ \dim Y = n\} \ . \tag{2}$$

In particular, for a real or complex Hilbert space H of dimension at least n, we always have $c_n(H) = c_n(\mathbb{K}^n)$.

Benítez, Sarantopoulos and Tonge [5] proved that for isomorphic Banach spaces X and Y we have $c_n(X) \le d^n(X,Y)c_n(Y)$, where $d(X,Y)$ denotes the Banach-Mazur distance of X and Y. Note, that for any n-dimensional space X a result of John [6] states that $d(X,\mathbb{K}^n) \le \sqrt{n}$ (where $\mathbb{K}^n$ denotes the n-dimensional Hilbert space). The combination of these results mean that the determination of $c_n(\mathbb{K}^n)$ gives information on the linear polarization constants of other spaces, too.

In this paper we are going to focus our attention to Hilbert spaces. Pappas and Révész [10] showed that $c(\mathbb{K}^n) = e^{-L(n,\mathbb{K})}$, where

$$L(n,\mathbb{K}) := \int_S \log|\langle \mathbf{x}, \mathbf{e}\rangle| d\sigma(\mathbf{x});$$

here S and σ denote the unit sphere and the normalized surface measure, respectively, and $\mathbf{e} \in S$ is an arbitrary unit vector. This result gives information on the asymptotic behaviour of $c_m(\mathbb{K}^n)$ as $m \to \infty$. However, the exact values of $c_m(\mathbb{K}^n)$ seem, in general, hopeless to determine.

A remarkable result of Arias-de-Reyna [1] states that $c_n(\mathbb{C}^n) = n^{n/2}$. Ball's recent solution [3] of the complex plank problem also implies the same result.

Compared to the complex case, the value of $c_n(\mathbb{R}^n)$ seems harder to find. The determination of $c_n(\mathbb{R}^n)$, by the definition and the Riesz representation theorem, boils down to determining

$$I := \inf_{\mathbf{x}_1,\ldots,\mathbf{x}_n\in S} \sup_{\|\mathbf{y}\|=1} |\langle \mathbf{x}_1, \mathbf{y}\rangle \cdots \langle \mathbf{x}_n, \mathbf{y}\rangle|$$

The estimate $I \le n^{-\frac{n}{2}}$ follows by considering an orthonormal system.

The complex result of Arias-de-Reyna can be used to derive the following estimates (see [11], where the argument is based on an interesting complexification result of [9]):

$$n^{\frac{n}{2}} \le c_n(\mathbb{R}^n) \le 2^{\frac{n}{2}-1} n^{\frac{n}{2}}. \tag{3}$$

A natural, intriguing conjecture, see [5], [11] is the following.

Conjecture. $c_n(\mathbb{R}^n) = n^{n/2}$.

Marcus (communicated in [7], and elaborated later in [11]) gives the following estimate: If $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ are unit vectors in $\mathbb{R}^n$ then there exists a unit vector $\mathbf{y}$ such that

$$|\langle \mathbf{x}_1, \mathbf{y}\rangle \cdots \langle \mathbf{x}_n, \mathbf{y}\rangle| \ge (\lambda_1/n)^{n/2}, \tag{4}$$

where λ_1 denotes the smallest eigenvalue of the Gram matrix $XX^* = [\langle \mathbf{x}_i, \mathbf{x}_j\rangle]$. Marcus also expressed the opinion that lower bounds on $\sup_{\|\mathbf{y}\|=1} |\langle \mathbf{x}_1, \mathbf{y}\rangle \cdots \langle \mathbf{x}_n, \mathbf{y}\rangle|$ should involve the eigenvalues $\lambda_1, \ldots, \lambda_n$ of the Gram matrix $XX^* = [\langle \mathbf{x}_i, \mathbf{x}_j\rangle]$, i.e. we should look for estimates of the form $\sup_{\|\mathbf{y}\|=1} |\langle \mathbf{x}_1, \mathbf{y}\rangle \cdots \langle \mathbf{x}_n, \mathbf{y}\rangle| \ge f(\lambda_1, \ldots, \lambda_n) n^{-n/2}$. Note that $\sum_j \lambda_j = \mathrm{Tr}\, XX^* = n$. Therefore the above Conjecture can be formulated as

$$\sup_{\|\mathbf{y}\|=1} |\langle \mathbf{x}_1, \mathbf{y}\rangle \cdots \langle \mathbf{x}_n, \mathbf{y}\rangle| \ge 1 \cdot n^{-n/2} = \left(\frac{\lambda_1 + \cdots + \lambda_n}{n}\right)^{n/2} n^{-n/2}.$$

In [8] the author proved that Marcus' estimate (4) can be improved to

$$\sup_{\|\mathbf{y}\|=1} |\langle \mathbf{x}_1, \mathbf{y}\rangle \cdots \langle \mathbf{x}_n, \mathbf{y}\rangle| \ge \left(\frac{n}{\lambda_1^{-1} + \cdots + \lambda_n^{-1}}\right)^{n/2} n^{-n/2}.$$

In the next section we will improve this result by replacing the harmonic mean of the numbers $\lambda_1, \ldots, \lambda_n$ by the geometric mean. Also, in the course of the proof we use two 'geometrical' lemmas which may be of independent interest. The original Conjecture (invoving the arithmetic mean of the numbers $\lambda_1, \ldots, \lambda_n$), however, still remains open.

2. A geometric lower bound

For the sake of simplicity we introduce the following notations:

Let b_n denote the volume of the n-dimensional closed unit ball B^n (we will not need the explicit value of b_n). Also, let $H_\alpha := \{\mathbf{z} = (z_1, \dots, z_n) \in \mathbb{R}^n : \ |\prod_{j=1}^n z_j| \geq \alpha \cdot n^{-n/2}\}$.

In order to prove our main result, Theorem 2.3, we will need the following two geometrical lemmas:

Lemma 2.1. *Let E be an n-dimensional ellipsoid symmetric with respect to the origin (i.e. the image of the n-dimensional unit ball under a linear transformation of full rank) of volume Vb_n. Assume that the $n-1$-dimensional 'horizontal slice' $E_0 := \{\mathbf{z} = (z_1, z_2, \dots z_n) \in E : z_n = 0\}$ has $n-1$-dimensional volume Sb_{n-1}. Then the horizontal slice at height h, $E_h := \{\mathbf{z} = (z_1, z_2, \dots z_n) \in E : \ z_n = h\}$ has $n-1$-dimensional volume*

$$f(V,S,h) = \begin{cases} (1 - (\frac{S}{V}h)^2)^{\frac{n-1}{2}} b_{n-1} & \text{if } |h| \leq V/S \\ 0 & \text{if } |h| > V/S \end{cases}$$

Proof. The essence of the lemma is that the function f depends only on V, S and h and not on the actual 'shape' of the ellipsoid.

The statement of the lemma is clear if E is a 'circular ellipsoid' whose axes are the same as the coordinate axes, i.e. E is the image of the unit ball B^n under the diagonal transformation

$$T := \begin{pmatrix} S^{1/(n-1)} & & & & 0 \\ & \cdot & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & S^{1/(n-1)} & \\ 0 & & & & \frac{V}{S} \end{pmatrix}$$

In the general case, let $E = A[B^n]$ be the image of the unit ball B^n under some transformation A, and assume that it posesses the prescribed parameters V, S, and let the height h also be given. The natural idea of the proof is that we transform the ellipsoid E to a circular ellipsoid whose axes are the coordinate axes and whose parameters are the same.

Let $\mathbf{r} := (r_1, \dots, r_n) \in E$ denote the point of E whose last coordinate r_n is maximal among the points of E, and let $\mathbf{q} \in B^n$ be its inverse image, i.e. $\mathbf{q} = A^{-1}\mathbf{r}$. Let $L_0 := A^{-1}[E_0]$.

Note that the $n-1$-dimensional L_0 is orthogonal to the vector $\mathbf{q}$, therefore there exists a unitary transformation U which takes the horizontal slice B_0^n of B^n to L_0 and the vertical unit vector $\mathbf{e}_n$ to $\mathbf{q}$ (note that if $n-1>1$ then U is not uniquely determined). Then we have $AU[B^n]=E$, $AU[B_0^n]=E_0$ and $AU(\mathbf{e}_n)=\mathbf{r}$.

The transformation AU maps the horizontal $n-1$-dimensional hyperplane $P_0:=\{\mathbf{x}=(x_1,\dots,x_n)\in\mathbb{R}^n:\ x_n=0\}$ onto itself. Denote the restriction of AU to P_0 by C_0. Now, take the $n-1$-dimensional transformation $\tilde{C}_1:=S^{1/(n-1)}C_0^{-1}$ of the horizontal hyperplane P_0. This preserves $n-1$-dimensional volume (i.e. it has determinant ±1) and takes the horizontal slice E_0 to $S^{1/(n-1)}B_0^n$.

Consider now the n-dimensional transformation

$$C_1:=\begin{pmatrix}\tilde{C}_1 & \mathbf{0}\\ \mathbf{0}^T & 1\end{pmatrix}.$$

It is clear that applying this transformation to E the image ellipsoid $C_1[E]$ will posses the same parameters as E, i.e. the same volume V, the same $n-1$-dimensional volume S of its horizontal slice $(C_1[E])_0=S^{1/n-1}B_0^n$, and the image $C_1[E_h]$ will still be at height h and have the same $n-1$-dimensional volume as E_h.

Let $\mathbf{s}:=(s_1,\dots,s_n)=C_1\mathbf{r}$.

Next we consider the transformation

$$C_2:=\begin{pmatrix}1 & & & 0 & -s_1/s_n\\ & \cdot & & & \cdot\\ & & \cdot & & \cdot\\ & & & \cdot & \cdot\\ & & & 1 & -s_{n-1}/s_n\\ 0 & & & 0 & 1\end{pmatrix}$$

Once again it is clear that the image $C_2C_1[E]$ has the same parameters V, S as E, and $(C_2C_1[E])_h=C_2C_1[E_h]$ with equal $n-1$-dimensional volume. To finish the proof it is enough to observe that $C_2C_1[E]=C_2C_1AU[B^n]=T[B^n]$ with the diagonal transformation T above. $\square$

The next lemma establishes the connection between ellipsoids and products of functionals.

Lemma 2.2. *Assume E is an n-dimensional ellipsoid of volume Vb_n (not necessarily centered at the origin). Then $E\cap H_V\neq\emptyset$.*

Proof. The proof proceeds by induction with respect to n.

For $n=1$ the statement is clear.

For an arbitrary n let $\mathbf{c}:=(c_1,\dots c_n)$ denote the centre of E, and assume, without loss of generality, that $c_n\geq0$. Let Sb_{n-1} be the $n-1$ dimensional volume of the horizontal slice $E\cap P_{c_n}$, where $P_{c_n}:=\{\mathbf{x}=(x_1,\dots,x_n)\in\mathbb{R}^n:\ x_n=c_n\}$. Now, let $h:=\frac{V}{S\sqrt{n}}$ and consider the horizontal hyperplane $P := P_{c_n+h}$. P is an $n-1$-dimensional space, and $P \cap H_V = \{\mathbf{z} = (z_1, \dots z_{n-1}) : |\prod_{j=1}^{n-1} z_j| \geq \frac{V}{c_n+h} n^{-n/2}\} \supset \{\mathbf{z} = (z_1, \dots z_{n-1}) : |\prod_{j=1}^{n-1} z_j| \geq \frac{V}{h} n^{-n/2}\}$. Furthermore, the $n-1$-dimensional volume of $P \cap E$ is $(1-\frac{1}{n})^{\frac{n-1}{2}} S b_{n-1}$ in view of Lemma 2.1 and the choice of h.

Finally, observe that $\frac{V}{h} n^{-n/2} = (n-1)^{\frac{-n+1}{2}} \left(\frac{V}{h}(1-\frac{1}{n})^{\frac{n-1}{2}} \frac{1}{\sqrt{n}} \right)$ and $(1-\frac{1}{n})^{\frac{n-1}{2}} S = \frac{V}{h}(1-\frac{1}{n})^{\frac{n-1}{2}} \frac{1}{\sqrt{n}}$, therefore the inductive hypothesis applies. $\square$

We are now in position to prove our new estimate on the norm of product of functionals.

Theorem 2.3. *Let unit vectors $\mathbf{x}_1, \dots, \mathbf{x}_n$ be given in $\mathbb{R}^n$, and let $\lambda_1, \dots, \lambda_n$ denote the eigenvalues of the Gram matrix $XX^* = [\langle \mathbf{x}_i, \mathbf{x}_j \rangle]$ (the matrix X is formed by the given vectors as rows). Then*

$$\sup_{\|\mathbf{y}\|=1} |\langle \mathbf{x}_1, \mathbf{y} \rangle \cdots \langle \mathbf{x}_n, \mathbf{y} \rangle| \geq \left(\prod_{j=1}^{n} \lambda_j \right)^{1/2} \cdot n^{-n/2} \tag{5}$$

Proof. We may assume that the vectors $\mathbf{x}_1, \mathbf{x}_2, \dots \mathbf{x}_n$ are linearly independent, otherwise the right hand side of the inequality is 0, and the estimate is meaningless. (We remark that other considerations , such as the ones in [8], also show that if we find a way to prove a good estimate in the case of linearly dependent vectors then we may get close to proving the original Conjecture. However, at present, there seems to be no better estimate than (3) for the linearly dependent case.)

The image E of the unit ball B^n under the transformation X is an n-dimensional ellipsoid of volume $V = \left(\prod_{j=1}^{n} \lambda_j \right)^{1/2} b_n$, therefore Lemma 2.2 gives the required result.

$\square$

Finally, let us make the following remarks.

An advantage of the proof applied above is that it is constructive in the sense that following the constructions of Lemma 2.2 we can actually find a vector $\mathbf{y}$ which satisfies (5). It is clear, however, that the estimate (5) does not settle the original Conjecture.

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