Trotter's product formula for projections

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11 February, 2002

Abstract

The aim of this paper is to examine the convergence of Trotter's product formula when one of the C_0-semigroups is replaced by a projection (which can always be regarded as a constant degenerate semigroup). The motivaton to study Trotter's formula in this setting arises from the fact that for 'nice' open sets $\Omega \subset \mathbb{R}^n$ the C_0-semigroup on $L^2(\Omega)$ generated by the Laplacian with Dirichlet boundary conditions can be obtained as a limit of a formula of this type.

Mathematics subject classification (2000): 47A05, 47D06

1 Introduction

Let A be the generator of a C_0-semigroup $(e^{tA})_{t\geq 0}$ on a Banach space E, and let $B \in \mathcal{L}(E)$. Then $A+B$ generates a C_0 semigroup which is given by Trotter's product formula

$$e^{t(A+B)} = \lim_{n\to\infty} (e^{\frac{t}{n}A}e^{\frac{t}{n}B})^n \tag{1}$$

where the limit is taken in the strong operator topology. A possible direction of generalization of this well-known result is discussed in [1] and [3]. Namely, the convergence of Trotter's product formula is examined in the case when the C_0-semigroup e^{tB} is replaced by the simplest of degenerate semigroups, i.e. a projection $P \in \mathcal{L}(E)$. For convenience we include the basic notions here:

A family of operators $S(t)_{t>0}$ is called a *semigroup* on E if $S : (0,\infty) \to \mathcal{L}(E)$ is strongly continuous and satisfies the semigroup property $S(t+s) = S(t)S(s)$ for all $s,t > 0$. If, in addition, $S(0) := \lim_{t\to 0} S(t)$ exists strongly, then we say that $S(t)_{t>0}$ (or $S(t)_{t\geq 0}$) is a *continuous degenerate semigroup*. In this case $S(0)$ is a bounded projection, its image $E_0 := S(0)E$ is invariant under $S(t)$ $(t \geq 0)$, and the restriction of $S(t)_{t\geq 0}$ to E_0 is a C_0-semigroup on E_0 and $S(t)$ equals 0 on $E_1 := (I - S(0))E$ (see [6], Theorem

*Suuported by the Marie Curie Host Fellowship "Transform Methods for Evolution Equations" at the University of Ulm.

†Supported by the student part of the NSF grant DMS-9800027.

10.5.5). A trivial example of a continuous degenerate semigroup is given by $S(t) := P$ $(t > 0)$, where P denotes a bounded projection.

Now, in (1) we replace the C_0-semigroup e^{tB} by the continuous degenerate semigroup $S(t) = P$ $(t > 0)$, and we examine the convergence of the formula

$$\lim_{n\to\infty} (e^{\frac{t}{n}A}P)^n \tag{2}$$

under various assumptions on A and P. (If (2) converges, then the limit can be regarded, in a sense, as the 'restriction' of the semigroup e^{tA} to the subspace PE. Of course, in the trivial case when e^{tA} and P commute, the formula (2) does converge to the restriction of e^{tA} to PE.) In Section 2 we describe some interesting conditions under which (2) converges strongly. For example, if A is the generator of the Gaussian semigroup on $L^2(\mathbb{R}^n)$ and $Pf = 1_\Omega f$ where $\Omega \subset \mathbb{R}^n$ is a bounded open domain with Lipschitz boundary, we will see that (2) converges strongly to the semigroup generated by the Dirichlet Laplacian on $L^2(\Omega)$. In Section 3 we provide some non-trivial examples where (2) fails to converge.

2 Convergence results

2.1 Bounded generators

The easiest case to study is, of course, that of bounded generators.

Theorem 1 *Let $A \in \mathcal{L}(E)$ be the generator of a C_0-semigroup $(e^{tA})_{t\geq 0}$ and let $P \in \mathcal{L}(E)$ be a projection. Then*

$$\lim_{n\to\infty} (e^{\frac{t}{n}A}P)^n x = e^{PAPt}Px$$

for all $x \in E$ and uniformly for $t \in [0, T]$.

Proof. Case 1. Assume first that both e^{tA} and P are contractive. Let $V(t) := Pe^{tA}P \in \mathcal{L}(PE)$ and apply Chernoff's product formula (see eg. [5], Theorem III.5.2) to the family $V(t)$ on the space PE. Note that $V(0) = I_{PE}$, $\|V(t)\| \leq 1$ (for all $t \geq 1$), and $\lim_{h\to 0} \frac{V(h)x_1 - x_1}{h} = PAx_1 = PAPx_1$ for all $x_1 \in PE$, and PAP is a bounded operator on PE. Now, by Chernoff's product formula $\lim_{n\to\infty}[V(\frac{t}{n})]^n x_1 = e^{PAPt}x_1$ for all $x_1 \in PE$ and uniformly for $t \in [0, T]$. Furthermore, for any given $x \in E$ we can decompose x as $x = Px + (I - P)x =: x_1 + x_2$ and we have $(e^{\frac{t}{n}A}P)^n x = (e^{\frac{t}{n}A}P)^n x_1 = e^{\frac{t}{n}A}(Pe^{\frac{t}{n}A}P)^{n-1}x_1$. Now, for large n we have

$$\|e^{PAPt}Px - (Pe^{\frac{t}{n}A}P)^n x_1\| = \|e^{PAPt}x_1 - (Pe^{\frac{t}{n}A}P)^n x_1\| < \varepsilon$$

for $t \in [0, T]$, and also

$$\|e^{\frac{t}{n}A}(Pe^{\frac{t}{n}A}P)^{n-1}x_1 - (Pe^{\frac{t}{n}A}P)^n x_1\| = \|(I-P)e^{\frac{t}{n}A}(Pe^{\frac{t}{n}A}P)^{n-1}x_1\| =$$
$$\|(I-P)(e^{\frac{t}{n}A} - I)(Pe^{\frac{t}{n}A}P)^{n-1}x_1\| \leq \|I-P\| \cdot \|e^{\frac{t}{n}A} - I\| \cdot \|x_1\| < \varepsilon$$

Case 2. In the general case we first introduce an equvivalent norm on E such that P becomes contractive, then we use a rescaling argument to achieve that the semigroup becomes contractive. Indeed, with the new norm $\|x\|_0 := \|Px\| + \|(I-P)x\|$ E is a Banach space, $\|\cdot\|$ and $\|\cdot\|_0$ are equivalent, and P is contractive on $E_{\|\cdot\|_0}$. Now, for $\lambda > \|A\|_0$ the rescaled semigroup $e^{-\lambda t}e^{At}$ is contractive on $E_{\|\cdot\|_0}$, therefore the result of Case 1 can be applied, and the result follows.

■

Remark 1. By similar arguments one can prove the following statement: if $(e^{tA})_{t\geq 0}$ is a C_0-semigroup on E and P is a finite dimensional projection with $\operatorname{Ran} P \subset D(A)$ then $\lim_{n\to\infty}(e^{\frac{t}{n}A}P)^n x = e^{PAPt}Px$ where e^{PAPt} is meant to be the C_0-semigroup on PE generated by the *bounded* operator PAP. See also Remark 4 below.

2.2 Positive semigroups

The results in this subsection are taken from [1].

Let (X, Σ, μ) be σ-finite measure space and let $(e^{tA})_{t\geq 0}$ be a positive C_0-semigroup on $E = L^p(X)$ where $1 \leq p < \infty$. Let $\Omega \subset X$ be measureable. Then $Pf := \mathbf{1}_\Omega f$ defines a projection on E, where $\mathbf{1}_\Omega$ denotes the characteristic function of Ω. In this subsection we will use the notation $L^p(\Omega)$ both in the usual sense and and in the sense to denote the subspace of functions f in $L^p(X)$ such that $f = 0$ almost everywhere in Ω^c. When a function f is in $L^p(\Omega)$ in the usual sense, we define the extension $\overline{f}$ on X by $\overline{f}|_\Omega = f$ and $\overline{f}|_{\Omega^c} = 0$. The following result holds (see [1], Theorem 5.3):

Theorem 2 *Let $f \in E$ and $t > 0$. Then*

$$S(t)f := \lim_{n\to\infty}(e^{\frac{t}{n}A}P)^n f$$

exists and $S(t)_{t>0}$ is a continuous degenerate semigroup of positive operators. Furthermore, $S(0) := \lim_{t\to 0} S(t)$ is a projection of the the form $S(0)f = \mathbf{1}_Y f$ where $Y \subset \Omega$ is a measureable set.

The continuous degenerate semigroup $S(t)_{t>0}$ can also be characterized by the following maximality property (see [1], Theorem 5.1): Let $T(t)_{t>0}$ be any semigroup of positive operators on $L^p(X)$ which maps $L^p(X)$ to $L^p(\Omega)$ and for which $0 \leq T(t)f \leq e^{tA}f$ for $t > 0$ and $0 \leq f \in L^p(X)$. Then $T(t)f \leq S(t)f$.

With the notations of Theorem 2 it can occur that $Y = \emptyset$ and $S(t) = 0$ (see [1], Example 5.4). However, in the following important case $Y = \Omega$ holds (for a detailed discussion of this Example and the following Remark see [1], Section 5 and 7):

Example 1 *(The Dirichlet Laplacian) Let $p = 2$, $X = \mathbb{R}^n$ (with Lebesgue measure) and $A = \Delta$ the Laplacian on $L^2(\mathbb{R}^n)$. Let Ω be a bounded open set with Lipschitz boundary. Then (with the notations of Theorem 2) we have $Y = \Omega$ and $S(t)|_{L^2(\Omega)} = e^{t\Delta_\Omega}$ where Δ_Ω is the Dirichlet Laplacian on $L^2(\Omega)$, i.e. $D(\Delta_\Omega) = \{f \in H_0^1(\Omega) : \Delta f \in L^2(\Omega)\}$ and $\Delta_\Omega f = \Delta f$.*

Remark 2. For general open sets Ω we still have $Y = \Omega$ and $S(t)|_{L^2(\Omega)} = e^{t\tilde{\Delta}_\Omega}$ where $\tilde{\Delta}_\Omega$ denotes the pseudo-Dirichlet Laplacian on $L^2(\Omega)$, i.e. $\tilde{\Delta}_\Omega$ is associated with the following densely-defined closed positive form a on $L^2(\Omega)$: $D(a) = \{f \in L^2(\Omega) : \overline{f} \in H^1(\mathbb{R}^n)\}$ and $a(f,f) = \int_{\mathbb{R}^n} |\overline{f}|^2 + \sum_{j=1}^n \int_{\mathbb{R}^n} |D_j\overline{f}|^2 = \int_\Omega |f|^2 + \sum_{j=1}^n \int_{\mathbb{R}^n} |D_j\overline{f}|^2$ (this statement is a consequence of Theorem 4 below). This means that we have $\tilde{\Delta}_\Omega = \Delta_\Omega$ whenever $D(a) = H_0^1(\Omega)$. It is not an aim of this paper to describe sets Ω where this occurs, but in the Example above we take boundedness and Lipschitz boundary as simple sufficient conditions.

2.3 Closed forms

In this subsection we describe another important case when Trotter's product formula converges. The results in this subsection are direct consequences of [8, Theorem and Addendum]. We describe the basic notions briefly:

Let H be a Hilbert space and let

$$a : D(a) \times D(a) \to \mathbb{C}$$

be a sesquilinear mapping where $D(a)$, the domain of a, is a is a subspace of H. We assume that a is semibounded, i.e. that there exists $\lambda \in \mathbb{R}$ such that

$$\|u\|_a^2 := \text{Re } a(u,u) + \lambda(u,u)_H > 0$$

for all $u \in D(a)$, $u \neq 0$. Moreover, we assume that $a + \lambda$ is sectorial and closed, i.e., that $|\text{Im } a(u,u)| \leq M(\text{Re } a(u,u) + \lambda(u,u)_H)$ and $(D(a), \|\cdot\|_a)$ is complete. In short, we will call a a *closed form*. Let $K = \overline{D(a)}$ be the closure of $D(a)$ in H. Denote by A the operator on K associated with a, i.e.

$$D(A) = \{u \in D(a) : \exists v \in K \text{ such that } a(u,\phi) = (v,\phi)_H \text{ for all } \phi \in D(a)\}$$

and $Au = v$. Then $-A$ generates a C_0-semigroup e^{-tA} on K. Denote by Q the orthogonal projection on K. Now, define the operator e^{-ta} on H by

$$e^{-ta}x = e^{-tA}Qx, \quad x \in H, \ t \geq 0$$

Then e^{-ta} is a continuous degenerate semigroup on H. We call it the *degenerate semigroup generated by a on H*.

Now, let b be a second closed form on H. Define $a + b$ on H by $D(a+b) = D(a) \cap D(b)$ and $(a+b)(u,v) = a(u,v) + b(u,v)$. Then it is easy to see that $a + b$ is a closed form again. Now the following product formula holds (see [8, Theorem and Addendum]):

Theorem 3 *Let $x \in H$. Then*

$$e^{-t(a+b)}x = \lim_{n\to\infty} (e^{-\frac{t}{n}a}e^{-\frac{t}{n}b})^n x$$

for all $t > 0$.

Remark 3. In [8, Addendum] this theorem is stated only for densely defined, closed forms a and b but the proof applies to the non-densely defined case, as well.

Now, let P be an orthogonal projection. Define the form b by $D(b) = PH$ and $b(u, v) = 0$ for all $u, v \in PH$. Then $e^{-tb} = P$ for all $t \geq 0$. Therefore, as a corollary of Theorem 3 we have

Theorem 4 *For any orthogonal projection P and closed form a, the limit*

$$S(t)x = \lim_{n\to\infty} (e^{-\frac{t}{n}a}P)^n x$$

exists for all $x \in H$ and $t > 0$, and $S(t)_{t>0}$ is the continuous degenerate semigroup generated by the form $a|_{PH}$.

There is another possible way to formulate this result. Let $T(z)_{z\in\Sigma_\tau}$ be a holomorphic C_0-semigroup on H, defined on a sector $\Sigma_\tau := \{z \in \mathbb{C} : z \neq 0, |\arg z| < \tau\}$, $\tau \in (0, \frac{\pi}{2}]$. Assume that $\|(T(z)\| \leq 1$ for all $z \in \Sigma_\tau$. Then the generator A of $T(z)$ is associated with a densely defined, semibounded, closed form a (see [7], Chapters VI. and IX., and also [2], Theorem 1.2), so we have the following corollary (see [3] Theorem 4):

Corollary 1 *Let $-A$ be the generator of a holomorphic C_0-semigroup $(e^{-zA})_{z\in\Sigma_\tau}$ on a Hilbert space H, where $\tau \in (0, \frac{\pi}{2}]$, and assume that $\|e^{-zA}\| \leq 1$ for all $z \in \Sigma_\tau$. Let P be an orthogonal projection. Then*

$$S(t)x = \lim_{n\to\infty} (e^{-\frac{t}{n}A}P)^n x$$

exists for all $x \in H$ and $t > 0$, and $S(t)_{t>0}$ is a continuous degenerate semigroup on H.

3 Counterexamples

In view of the results in Section 1 one may conjecture that (2) converges in more general settings. In particular, the following conjectures were given in [3]:

(a) Let e^{tA} be a contractive C_0-semigroup on a Hilbert space H, and let P be an orthogonal projection. Then (2) should converge.

(b) Let e^{tA} be a positive, contractive C_0-semigroup on $L^p(X, \Sigma, \mu)$ (where (X, Σ, μ) is a σ-finite measrure space, and $1 < p < \infty$), and let P be a positive, contractive projection. Then (2) should converge.

In this section we present two examples which disprove these conjectures. We remark that the case $p = 1$ in conjecture (b) was not included, because a positive, contractive C_0-semigroup and a positive, contractive projection on $E = L^1([0, 1])$), such that (2) fails to converge, was already provided in [3].

3.1 Hilbert case

Let us remark that by using the theory of unitary dilations of contractive C_0-semigroups in Hilbert spaces (see e.g. [4], Corollary 6.14) one can reduce the first conjecture to the case of unitary C_0-semigroups. Therefore, we are looking for a counterexample among unitary C_0-semigroups instead of arbitrary contractive ones.

We carry out our construction in the space $L^2[0,1]$. As an example of unitary semigroup we take the semigroup of multiplications by e^{ith}, where h is a real-valued, measurable function on $[0,1]$, to be specified later. We choose P to be the one-dimensional orthogonal projection onto the space of constant functions, i.e. $Pf = \mathbf{1} \cdot \int_0^1 f(x)dx$. As a test function on which (2) will fail for $t=1$, we take $\mathbf{1}$.

Denoting $c_n = \int_0^1 e^{i\frac{1}{n}h(x)}dx$, the function $\left[e^{\frac{1}{n}A}P\right]^n(\mathbf{1})$ becomes $c_n^{n-1}e^{i\frac{1}{n}h}$. However, by the Lebesgue Dominated Convergence Theorem, $\lim_{n\to\infty} c_n = 1$ as well as $\lim_{n\to\infty} e^{i\frac{1}{n}h} = \mathbf{1}$ in $L_2[0,1]$. So, $\lim_{n\to\infty}\left[e^{\frac{1}{n}A}P\right]^n(\mathbf{1})$ exists in $L^2[0,1]$ if and only if the numerical limit

$$\lim_{n\to\infty} c_n^n \tag{3}$$

exists. Now we specify the function h, for which we prove that (3) diverges. Put $h = \sum_{k=1}^{\infty} \chi_{(1/2^k,1/2^{k-1}]}2^k\pi$. Then $c_n = \sum_{k=1}^{\infty}\frac{1}{2^k}e^{i\frac{1}{n}2^k\pi}$. We show the following two inequalities

$$\liminf_{n\to\infty} |c_{2^n}|^{2^n} \geq e^{-(4+\frac{\pi^2}{4})} \tag{4}$$

$$\limsup_{n\to\infty} |c_{2^n 3}|^{2^n 3} \leq e^{-(6+\frac{\pi^2}{6}-\frac{\pi^4}{27\cdot 24\cdot 7})}. \tag{5}$$

Noticing that $4+\frac{\pi^2}{4} < 6+\frac{\pi^2}{6}-\frac{\pi^4}{27\cdot 24\cdot 7}$ we get the desired result.

Let us show (4) first. Observe that

$$c_{2^n} = \sum_{k=1}^{n-1}\frac{1}{2^k}e^{i\frac{2^k}{2^n}\pi} - \frac{1}{2^n} + \sum_{k=n+1}^{\infty}\frac{1}{2^k} = \sum_{k=1}^{n-1}\frac{1}{2^k}e^{i\frac{2^k}{2^n}\pi}.$$

Using the inequality $\cos(\alpha) \geq 1-\frac{\alpha^2}{2}$ we get

$$\begin{aligned}|c_{2^n}| \geq |\mathrm{Re}\ c_{2^n}| &= \sum_{k=1}^{n-2}\frac{1}{2^k}\cos\left(\frac{2^k}{2^n}\pi\right) \geq \sum_{k=1}^{n-2}\frac{1}{2^k}\left(1-\frac{\pi^2}{2}\frac{4^k}{4^n}\right)\\ &= 1-\frac{4}{2^n}-\frac{\pi^2}{2}\frac{1}{4^n}(2^{n-1}-2) = 1-\frac{1}{2^n}\left(4+\frac{\pi^2}{4}\right)+\frac{\pi^2}{4^n}.\end{aligned}$$

Since $\lim_{N\to\infty}(1+\frac{a}{N}+\frac{b}{N^2})^N = e^a$, we obtain (4).

To prove (5) let us simplify c_{2^n3}. We have

$$
\begin{aligned}
c_{2^n3} &= \sum_{k=1}^{n-1} \frac{1}{2^k} e^{i\frac{2^k}{2^n3}\pi} + \frac{1}{2^n} e^{i\frac{1}{3}\pi} + \sum_{k=n+1}^{\infty} \frac{1}{2^k} e^{i\frac{2^{k-n}}{3}\pi} \\
&= \sum_{k=1}^{n-1} \frac{1}{2^k} e^{i\frac{2^k}{2^n3}\pi} + \frac{1}{2^n}(\frac{1}{2} + i\frac{\sqrt{3}}{2}) + \frac{1}{2^n} \sum_{k=1}^{\infty} \frac{1}{2^k} e^{i\frac{2^k}{3}\pi}.
\end{aligned}
$$

Notice that $e^{i\frac{2^k}{3}\pi} = e^{i(-1)^{k+1}\frac{2}{3}\pi} = -\frac{1}{2} + i(-1)^{k+1}\frac{\sqrt{3}}{2}$. Thus, $\sum_{k=1}^{\infty} \frac{1}{2^k} e^{i\frac{2^k}{3}\pi} = -\frac{1}{2} + i\frac{\sqrt{3}}{6}$. After these computations c_{2^n3} becomes

$$
\sum_{k=1}^{n-1} \frac{1}{2^k} e^{i\frac{2^k}{2^n3}\pi} + i\frac{2\sqrt{3}}{2^n3}.
$$

Now using the inequality $\cos(\alpha) \le 1 - \frac{\alpha^2}{2} + \frac{\alpha^4}{24}$ we obtain the following estimate

$$
\begin{aligned}
|\mathrm{Re}\ c_{2^n3}| &\le \sum_{k=1}^{n-1} \frac{1}{2^k} \left(1 - \frac{\pi^2}{18}\frac{4^k}{4^n} + \frac{\pi^4}{81 \cdot 24}\frac{16^k}{16^n}\right) \\
&= 1 - \frac{1}{2^{n-1}} - \frac{\pi^2}{18}\frac{2^n - 2}{4^n} + \frac{\pi^4}{81 \cdot 24}\frac{8^n - 8}{16^n 7} \\
&= 1 - \frac{1}{2^n3}\left(6 + \frac{\pi^2}{6} - \frac{\pi^4}{27 \cdot 24 \cdot 7}\right) + \frac{a}{(2^n3)^2} + \frac{b}{(2^n3)^4},
\end{aligned}
$$

for some constants a and b. Similarly, using $\sin(\alpha) \le \alpha$, we have

$$
|\mathrm{Im}\ c_{2^n3}| \le \sum_{k=1}^{n-1} \frac{1}{2^k}\frac{2^k}{2^n3}\pi + \frac{2\sqrt{3}}{2^n3} \le \frac{(n+1)\pi}{2^n3}.
$$

Thus,

$$
\begin{aligned}
|c_{2^n3}|^{2^n3} &= \left(|\mathrm{Re}\ c_{2^n3}|^2 + |\mathrm{Im}\ c_{2^n3}|^2\right)^{\frac{2^n3}{2}} \\
&\le (1 - \frac{2}{2^n3}(6 + \frac{\pi^2}{6} - \frac{\pi^4}{27 \cdot 24 \cdot 7}) + (\frac{2}{2^n3})^2(n+1)^2 a_1 \\
&+ (\frac{2}{2^n3})^2 a_2 + \ldots + (\frac{2}{2^n3})^8 a_8)^{\frac{2^n3}{2}}.
\end{aligned}
$$

Passing to the upper limit as $n \to \infty$, we finally obtain (5).

Remark 4. The function $\mathbf{1}$ is not in the domain of the generator A of our semigroup. In fact, we see from Remark 1 above that for any function $f \in D(A)$, $\|f\| = 1$ the formula (2) converges and we have

$$
\lim_{n\to\infty} (e^{\frac{t}{n}A} P_f)^n f = e^{(Af,f)} \cdot f
$$

where P_f denotes the orthogonal projection on the 1-dimensional subspace spanned by f.

3.2 L^p-case for positive semigroups

Our second example is on the Hilbert space $L^2[0,2\pi]$, but now for a positive contractive C_0-semigroup and positive contractive projection.

We take $e^{tA}f(x) = f(x+2\pi t)$, regarding f as a 2π-periodic function. Now let P be the orthogonal projection onto the space spanned by the positive norm-one function $g(x) = \frac{1}{\sqrt{34\pi}}\left[4 + \sum_{k=0}^{\infty} \frac{1}{\sqrt{2^k}} \cos 2^k x\right]$. Notice that, like in the previous example, our projection is one-dimensional (see Remark 5 below). Simple substitution shows that (2) evaluated at g for $t=1$ exists if and only if the numerical limit $\lim_{n\to\infty}\left[\int_0^{2\pi} g(x)g(x+\frac{1}{n})dx\right]^n$ exists. Denoting

$$c_n = \int\limits_0^{2\pi} g(x)g(x+\frac{1}{n})dx$$

and using the orthogonality of cosines, we obtain

$$c_n = \frac{16}{17} + \frac{1}{17}\sum_{k=1}^{\infty}\frac{1}{2^k}\cos\frac{2^k}{n}\pi$$

Following the same calculations as for the first example, we obtain inequalities (4) and (5) with powers doubled on the right hand sides.

This disproves the second conjecture.

Remark 5. As we have already noticed, the projections in our examples are one-dimensional. It would be interesting to know what property of a C_0-semigroup on a Hilbert space is responsible for the existence of (2) for all one-dimensional, or more specifically, one-dimensional orthogonal projections.

The authors are grateful to Wolfgang Arendt, András Bátkai and Bálint Farkas for helpful conversations.

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