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# Recognizing linear structure in noisy matrices☆

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**Abstract**

Behaviour of the eigenvalues of random matrices with an underlying linear structure is investigated, when the structure is exposed to random noise. The question, how a deterministic skeleton behind a random matrix can be recognized, is also discussed. Such random matrices, as weight matrices of random graphs, adequately describe some large biological and communication networks. A range for the power of random power law graphs—for which the structure is robust enough—is established.


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## 1. Introduction

Mostly we think of random matrices as completely random Wigner-type matrices whose eigenvalues obey the semi-circle law. No matter how important this type of a matrix in quantum mechanics was, in case of real-life matrices it is merely a random noise added to the underlying linear structure of the matrix (if there is any).

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Although, it is hard to recognize the structure concealed by the noise, in a number of models it is possible by means of spectral techniques and large deviations principles.

Usually, our matrix is the weight matrix of some random weighted graph $G = (V, \mathbf{A})$ with an $n$-element vertex set $V$ and $n \times n$ symmetric weight matrix $\mathbf{A}$, where $n \to \infty$. For example, some communication, social or biological networks can be adequately described by a random graph model. Performing graph-embedding techniques, it is a crucial question how many protruding eigenvalues—with corresponding eigenvectors—to choose for the vertex-representation.

Also, the classical numerical algorithms for the spectral decomposition of a matrix with size exceeding a million are not immediately applicable, and some newly developed randomized algorithms are to be used instead, see [1]. These algorithms exploit the randomness of our matrix, and rely on the fact that a random noise will not change the order of magnitude of the relevant eigenvalues with large absolute value. Sometimes—instead of depriving our matrix of the noise—a noise is added (by digitalizing the entries of or making the underlying matrix sparse by an appropriate randomization) to make the matrix more easily decomposable by means of the classical methods. For example, the Lánczos method (see Section 9 of [11]) is applicable to large, sparse, symmetric eigenproblems.

Both the number of eigenvalues to be kept and algorithmic questions can be—at least partly—analyzed by means of the results in Sections 2 and 3. For an easy discussion, in [6] we introduced the notion of Wigner-noise that is a generalization of a random matrix investigated by Wigner [15] and the eigenvalues of which obey the semi-circle law (if the order of the matrix tends to infinity). We cite the definitions.

**Definition 1.1.** The $n \times n$ real matrix $\mathbf{W}$ is a *Wigner-noise* if it is symmetric, its entries $w_{ij}$, $1 \leqslant i \leqslant j \leqslant n$, are independent random variables, $\mathbb{E}(w_{ij}) = 0$, $\mathrm{Var}(w_{ij}) \leqslant \sigma^2$ with some $0 < \sigma < \infty$ and either the $w_{ij}$'s are uniformly bounded (there is a constant $K > 0$ such that $|w_{ij}| \leqslant K$) or they are Gaussian distributed.

For example, mutations in cellular networks as well as random effects in social networks can be modelled by a Wigner-noise. By the method of Füredi and Komlós [10] it can be proved (see [1]) that for the maximum absolute value eigenvalue of $\mathbf{W}$

$$\max_{1 \leqslant i \leqslant n} |\lambda_i(\mathbf{W})| \leqslant 2\sigma\sqrt{n} + \mathrm{O}(n^{1/3}\log n) \tag{1.1}$$

holds with probability tending to 1, if $n \to \infty$.

In the sequel, we put this noise on the following general deterministic structure.

**Definition 1.2.** The $n \times n$ matrix $\mathbf{B}$ is a *blown up matrix*, if there is a constant $k < n$, a $k \times k$ symmetric so-called *pattern matrix* $\mathbf{P}$ with entries $0 \leqslant p_{ij} \leqslant 1$, and there are positive integers $n_1, \ldots, n_k$, $\sum_{i=1}^{k} n_i = n$ such that $\mathbf{B}$ can be divided into $k^2$ blocks, the block $(i, j)$ being an $n_i \times n_j$ matrix with entries all equal to $p_{ij}$ $(1 \leqslant i, j \leqslant k)$.

In particular, if $n_1 = \ldots = n_k = n/k$, then $\mathbf{B} = \mathbf{P} \otimes \mathbf{F}$, where $\mathbf{F}$ is the $n/k \times n/k$ all 1's matrix and $\otimes$ is the Kronecker-product of matrices.

Now $k$ will be kept fixed, while $n_1, \ldots, n_k$ will tend to infinity in the same order, and we put a Wigner-noise on our blown up matrix.

**Definition 1.3.** Let $\mathbf{B}$ be a blown up matrix of Definition 1.2, and $\mathbf{W}$ be an $n \times n$ Wigner-noise of Definition 1.1. We say that the property $\mathscr{T}_n$ holds *almost surely* for the $n \times n$ random matrix $\mathbf{A} = \mathbf{B} + \mathbf{W}$, if the probability that $\mathbf{A}$ has the property $\mathscr{T}_n$ tends to 1, if $n \to \infty$ in such a way that $n_i/n \geqslant c$ with some constant $c$ for $i = 1, \ldots, k$.

Under the above conditions, in [5] we thoroughly investigated a special case of a block-matrix perturbed by Wigner-noise. Now similar results will be proved for a general blown up matrix $\mathbf{B}$. Under the notation of Definitions 1.1–1.3, in Section 2 we shall prove that $\mathbf{B} + \mathbf{W}$ will have almost surely $k$ protruding eigenvalues.

In the random graph setup, in Section 3 it will be shown that the $k$-dimensional Euclidean representation of the vertices—via eigenvectors corresponding to the protruding eigenvalues—also indicates the block structure. With an appropriate Wigner-noise our perturbed graph is a usual random graph with weights 1 or 0 (indicating the presence or absence of the corresponding edge with certain probability).

In summary, the Wigner-noise is sufficiently general to include a lot of random matrices as special cases of adding such a noise. However, I do not mean that this noise is negligible. In quantum mechanics it played an independent role, but if added to a matrix with an effective linear structure it is not able to destroy that structure. Probably, the Wigner-noise plays a similar role among random matrices, as the white noise (Wiener-process) plays among stochastic processes.

In Section 4 the reversed question is investigated: how can we find a blown up skeleton behind an arbitrary random matrix from everyday life? We shall prove that an $n \times n$ random matrix under very general conditions has at least one eigenvalue greater than $\sqrt{n}$ in magnitude. Suppose, there are $k$ eigenvalues of order greater than $\sqrt{n}$. If the so-called $k$-variance of the representatives of the vertices—by means of the corresponding eigenvectors—is "small enough", we also give a construction for a blown up structure. There are other approaches for clustering large graphs via the singular value decomposition, see [9].

Section 5 is about the existence of a deterministic structure behind a random graph, that is guaranteed by the Regularity Lemma of Szemerédi [13] under appropriate density conditions. Other kinds of random graphs are frequently investigated nowadays, like random power law graphs. Such so-called scale-free networks—developed by preferential attachment—are frequently used to model the graph of the internet, social connections, or metabolic networks of cells [3]. Let $\beta > 0$ denote the power in the distribution of the actual degrees of a random power law graph introduced in [8]: the probability that a vertex has degree $x$ is proportional to $1/x^{\beta}$. Here the skeleton is a diadic product, and we shall prove that such graphs burdened with a Wigner-noise are robust in the range of $1.5 \leqslant \beta < 2$. Cellular networks frequently are in this domain (see [2]), and—possibly just because of this—they can tolerate random noise (like mutations) very well.

## 2. Spectral properties of blown up weighted matrices

By the notation of Definition 1.2 let $\mathbf{B}$ be an $n \times n$ blown up matrix of the $k \times k$ symmetric pattern matrix $\mathbf{P}$. Let $V_1, \ldots, V_k$ denote the partition of the index set $\{1, \ldots, n\}$ with respect to the blow-up, $|V_i| = n_i$ $(i = 1, \ldots, k)$, $\sum_{i=1}^k n_i = n$.

**Proposition 2.1.** *All the non-zero eigenvalues of the $n \times n$ blown up matrix $\mathbf{B}$ are of order $n$ in absolute value.*

This statement is proved in [6]. To be self-contained, we include the proof, as its ideas will be applied in the proof of the next proposition.

**Proof.** As there are at most $k$ linearly independent rows in $\mathbf{B}$, the zero is an eigenvalue of it with multiplicity at least $n - k$. It can easily be seen that any eigenvector corresponding to a non-zero eigenvalue of $\mathbf{B}$ has equal coordinates within the blocks $V_1, \ldots, V_k$. Let $\mathbf{y}$ be such an eigenvector with $n_1$ coordinates being equal to $y_1, \ldots, n_k$ coordinates being equal to $y_k$, and $\beta$ be the corresponding eigenvalue. Then

$$\sum_{j=1}^{k} n_j p_{ij} y_j = \beta y_i \quad (i = 1, \ldots, k).$$

Observe that the same eigenvalue–eigenvector equation belongs to the matrix $\mathbf{PD} = n\mathbf{P}\widetilde{\mathbf{D}}$, where

$$\mathbf{D} = \operatorname{diag}(n_1, \ldots, n_k) \quad \text{and} \quad \widetilde{\mathbf{D}} = \operatorname{diag}\left(\frac{n_1}{n}, \ldots, \frac{n_k}{n}\right). \tag{2.1}$$

We remark that $\mathbf{PD}$ and $\mathbf{P}\widetilde{\mathbf{D}}$ are not symmetric matrices but—due to this coincidence of spectra—they also have real eigenvalues. Let $\gamma_1, \ldots, \gamma_r$ denote the non-zero eigenvalues of $\mathbf{P}\widetilde{\mathbf{D}}$, $r = \operatorname{rank}(\mathbf{P}\widetilde{\mathbf{D}}) \leqslant k$. As their absolute values are also the singular values of $\mathbf{P}\widetilde{\mathbf{D}}$,

$$0 < \min_{1\leqslant i\leqslant r} |\gamma_i| \leqslant \max_{1\leqslant i\leqslant r} |\gamma_i| \leqslant \max_{1\leqslant i\leqslant r} |\lambda_i(\mathbf{P})| \cdot \max_{1\leqslant i\leqslant r} |\lambda_i(\widetilde{\mathbf{D}})| \leqslant \max_{1\leqslant i\leqslant r} |\lambda_i(\mathbf{P})| \leqslant k$$

holds for $\gamma_i$'s, therefore the absolute values of the non-zero eigenvalues $\beta_i$'s ($\beta_i = n\gamma_i$) of $\mathbf{B}$ are of order $n$, that is

$$|\beta_i| = \Theta(n), \quad i = 1, \ldots, r. \tag{2.2}$$

If $\mathbf{PD}$, and hence, $\mathbf{P}\widetilde{\mathbf{D}}$ happens to be singular $(r < k)$, this fact results in additional zero eigenvalues of $\mathbf{B}$, but the non-zero eigenvalues are still of order $n$. □

We remark that the symmetric matrix $\widetilde{\mathbf{D}}^{1/2}\mathbf{P}\widetilde{\mathbf{D}}^{1/2}$ has the same eigenvalues—$\gamma_1, \ldots, \gamma_k$—as $\mathbf{P}\widetilde{\mathbf{D}}$, since $\widetilde{\mathbf{D}}^{1/2}\mathbf{P}\widetilde{\mathbf{D}}^{1/2}\mathbf{x} = \gamma\mathbf{x}$ is equivalent to $\mathbf{P}\widetilde{\mathbf{D}}(\widetilde{\mathbf{D}}^{-1/2}\mathbf{x}) = \gamma(\widetilde{\mathbf{D}}^{-1/2}\mathbf{x})$. Though, the corresponding eigenvectors of $\mathbf{P}\widetilde{\mathbf{D}}$ are not pairwise orthogonal.

In the following special case we can prove a little bit more:

**Proposition 2.2.** *Let the entries of the $k \times k$ pattern matrix be the following*: $p_{ii} = 0$ $(i = 1, \ldots, k)$ *and* $p_{ij} = p_{ji} = p \in [0, 1]$ $(1 \leqslant i < j \leqslant k)$. *Let* $\mathbf{B}$ *be the blown up matrix of* $\mathbf{P}$ *with block sizes* $n_1 \leqslant n_2 \leqslant \cdots \leqslant n_k$, $n := \sum_{i=1}^{k} n_i$. *Then* $\mathbf{B}$ *has exactly* $n - k$ *zero eigenvalues*, *the negative eigenvalues of* $\mathbf{B}$ *are in the interval* $[-pn_k, -pn_1]$, *while the positive ones in* $[p(n - n_k), p(n - n_1)]$.

**Proof.** It is sufficient to prove for $p = 1$. In the case $0 < p < 1$ the statement of the proposition follows from this, as the pattern matrix is multiplied by $p$, therefore, all the eigenvalues of $\mathbf{P}$ and consequently, those of $\mathbf{B}$ are also multiplied by $p$. In the trivial case $p = 0$ all the eigenvalues are zeroes.

For a general blown up matrix we have already seen that its rank is at most $k$. Now it is exactly $k$, as the rank of the matrix $\mathbf{PD}$ is exactly $k$. So zero is an eigenvalue of $\mathbf{B}$ with multiplicity $n - k$ and corresponding eigenspace

$$\left\{ \mathbf{x} = (x_1, \ldots, x_n) : \sum_{j \in V_i} x_j = 0,\ i = 1, \ldots, k;\ \mathbf{x} \neq 0 \right\} \subset \mathbb{R}^n.$$

Due to the orthogonality, any eigenvector $\mathbf{y}$ belonging to an eigenvalue $\beta \neq 0$ of $\mathbf{B}$ has $n_1$ coordinates equal to $y_1, \ldots,$ and $n_k$ coordinates equal to $y_k$. The corresponding eigenvalue–eigenvector equation $\mathbf{By} = \beta\mathbf{y}$ gives that

$$\sum_{l \neq i} n_l y_l = \beta y_i \quad (i = 1, \ldots, k), \tag{2.3}$$

consequently

$$\sum_{l=1}^{k} n_l y_l = (n_i + \beta) y_i \quad (i = 1, \ldots, k), \tag{2.4}$$

that is—with regard to the left-hand side—independent of $i$.

If $\beta = -n_i$ for some index $i$ then $\beta$ is in the desired range, and there is nothing to prove. If $\beta_i \neq -n_i$ $(i = 1, \ldots, k)$ then none of the $y_i$'s can be zero (otherwise—due to (2.4)—all the $y_i$'s were zeroes, but the zero vector cannot be an eigenvector). Let $i$ be an arbitrary integer in $\{1, \ldots, k\}$. As $y_i \neq 0$, $\mathbf{y}$ can be scaled such that $y_i = 1$. Therefore (2.4) becomes

$$\sum_{l=1}^{k} n_l y_l = n_i + \beta. \tag{2.5}$$

Equating (2.5) with (2.4) applied for the other indices implies that

$$y_j = \frac{n_i + \beta}{n_j + \beta} \quad (j \neq i).$$

Summing up for $j = 1, \ldots, k$

$$\sum_{j=1}^{k} n_j y_j = (n_i + \beta) \sum_{j=1}^{k} \frac{n_j}{n_j + \beta}$$

follows, and by (2.5) it is also equal to $n_i + \beta$, therefore

$$\sum_{j=1}^{k} \frac{n_j}{n_j + \beta} = 1. \tag{2.6}$$

As $\operatorname{tr} \mathbf{B} = 0$, there must be both negative and positive eigenvalues of $\mathbf{B}$. Let us suppose that there is an eigenvalue $\beta < -n_k$. Then on the left-hand side of (2.6) all the terms were negative, and their sum could not be 1. Consequently, all the eigenvalues must be at least $-n_k$. Now let us suppose that there is a negative eigenvalue with $-n_1 < \beta < 0$. Then for all the terms on the left-hand side of (2.6)

$$\frac{n_j}{n_j + \beta} > 1 \quad (j = 1, \ldots, k)$$

holds, therefore their sum cannot be 1. So, for the negative eigenvalues

$$-n_k \leqslant \beta \leqslant -n_1$$

is proved.

For the positive eigenvalues we shall use that

$$0 < n_1 + \beta \leqslant n_j + \beta \leqslant n_k + \beta \quad (j = 1, \ldots, k).$$

Taking the reciprocals, multiplying by $n_j$, and summing up for $j = 1, \ldots, k$ we obtain that

$$\sum_{j=1}^{k} \frac{n_j}{n_1 + \beta} \geqslant \sum_{j=1}^{k} \frac{n_j}{n_j + \beta} \geqslant \sum_{j=1}^{k} \frac{n_j}{n_k + \beta},$$

that is, in view of (2.6),

$$\frac{n}{n_1 + \beta} \geqslant 1 \geqslant \frac{n}{n_k + \beta},$$

which implies

$$n - n_k \leqslant \beta \leqslant n - n_1,$$

that was to be proved for the positive eigenvalues in the case of $p = 1$. □

**Remarks**

1. In the special case $n_1 = \ldots = n_k = n/k$ all the negative eigenvalues of **B** are equal to $-pn/k$, and all the positive ones to $p(n - n/k)$. As the sum of the eigenvalues of **B** is zero, $-pn/k$ is an eigenvalue with multiplicity $k - 1$, while $p(n - n/k)$ is a single eigenvalue.
2. If $n_i$ is a block-size with multiplicity $k_i$ ($\sum_{i=1}^{k} k_i = k$) then $-pn_i$ is an eigenvalue of **B** with multiplicity $k_i - 1$. Accordingly, if $n_i$ is a single block-size then $-pn_i$ cannot be an eigenvalue of **B**. If especially $k_1 = k$ then $-pn/k$ is an eigenvalue with multiplicity $k - 1$, in accordance with the previous remark.
3. In the case $p = 1$ our matrix **B** is the adjacency matrix of $K_{n_1,\ldots,n_k}$, the complete $k$-partite graph on disjoint, edge-free vertex sets $V_1, \ldots, V_k$ with $|V_i| = n_i$ ($i = 1, \ldots, k$).

**Theorem 2.3.** *Let* **B** *be a blown up matrix of Definition 1.2 with non-zero eigenvalues* $\beta_1, \ldots, \beta_r$ ($r \leqslant k$), *and* **W** *be an* $n \times n$ *Wigner-noise. Then there are* $r$ *eigenvalues* $\lambda_1, \ldots, \lambda_r$ *of the noisy random matrix* $\mathbf{A} = \mathbf{B} + \mathbf{W}$ *such that*

$$|\lambda_i - \beta_i| \leqslant 2\sigma\sqrt{n} + \mathrm{O}(n^{1/3}\log n), \quad i = 1, \ldots, r \tag{2.7}$$

*and for the other* $n - r$ *eigenvalues*

$$|\lambda_j| \leqslant 2\sigma\sqrt{n} + \mathrm{O}(n^{1/3}\log n), \quad j = r + 1, \ldots, n \tag{2.8}$$

*holds almost surely.*

**Proof.** The statement immediately follows by applying the Weyl's perturbation theorem [16] for the spectrum of **B** characterized in Proposition 2.1, where the spectral norm of the perturbation **W** is estimated by (1.1). □

Consequently, taking into account the order $\Theta(n)$ of the non-zero eigenvalues of **B**, there is a spectral gap between the $r$ largest absolute value and the other eigenvalues of **A**, this is of order $\Delta - 2\varepsilon$, where

$$\varepsilon := 2\sigma\sqrt{n} + \mathrm{O}(n^{1/3}\log n) \quad \text{and} \quad \Delta := \min_{1\leqslant i\leqslant r} |\beta_i|. \tag{2.9}$$

In general, $r = \operatorname{rank} \mathbf{B} = k$, and Theorem 2.3 guarantees the existence of $k$ protruding eigenvalues of **A**.

## 3. Euclidean representation of blown up weighted graphs

Suppose that $\operatorname{rank} \mathbf{B} = k$. With the help of Theorem 2.3 we can also estimate the distances between the corresponding eigenspaces of the matrices **B** and $\mathbf{A} = \mathbf{B} + \mathbf{W}$. Let us denote the unit norm eigenvectors belonging to the eigenvalues $\beta_1, \ldots, \beta_k$ of **B** by $\mathbf{y}_1, \ldots, \mathbf{y}_k$ and those belonging to the eigenvalues $\lambda_1, \ldots, \lambda_k$ of **A** by $\mathbf{x}_1, \ldots, \mathbf{x}_k$.

Let $F := \text{Span}\,\{\mathbf{y}_1, \ldots, \mathbf{y}_k\} \subset \mathbb{R}^n$ be $k$-dimensional subset, and let $\text{dist}(\mathbf{x}, F)$ denote the Euclidean distance between the vector $\mathbf{x} \in \mathbb{R}^n$ and the subspace $F$.

**Proposition 3.1.** *With the above notation the following estimate holds almost surely for the sum of the squared distances between* $\mathbf{x}_1, \ldots, \mathbf{x}_k$ *and* $F$ :

$$\sum_{i=1}^{k} \text{dist}^2(\mathbf{x}_i, F) \leqslant k \frac{\varepsilon^2}{(\Delta - \varepsilon)^2} = \mathrm{O}\left(\frac{1}{n}\right), \tag{3.1}$$

*where the order of the estimate follows from the order of* $\varepsilon$ *and* $\Delta$ *of* (2.9).

**Proof.** Let us choose one of the eigenvectors $\mathbf{x}_1, \ldots, \mathbf{x}_k$ of $\mathbf{A} = \mathbf{B} + \mathbf{W}$ and denote it simply by $\mathbf{x}$ with corresponding eigenvalue $\lambda$. We shall estimate the distance between $\mathbf{x}$ and $F$. For this purpose we expand $\mathbf{x}$ in the basis $\mathbf{y}_1, \ldots, \mathbf{y}_n$ with coefficients $t_1, \ldots, t_n \in \mathbb{R}$:

$$\mathbf{x} = \sum_{i=1}^{n} t_i \mathbf{y}_i.$$

The eigenvalues of the matrix $\mathbf{B}$ corresponding to $\mathbf{y}_1, \ldots, \mathbf{y}_n$ are denoted by $\beta_1, \ldots, \beta_n$, where the $k$ largest eigenvalues $\beta_1, \ldots, \beta_k$ are those defined in the proof of Proposition 2.1 (we can assume that they are in non-increasing order with the proper reordering of the blocks), and there is a sudden drop following these eigenvalues in the spectrum of $\mathbf{B}$, as $\beta_{k+1} = \cdots = \beta_n = 0$. Then, on the one hand

$$\mathbf{A}\mathbf{x} = (\mathbf{B} + \mathbf{W})\mathbf{x} = \sum_{i=1}^{n} t_i \beta_i \mathbf{y}_i + \mathbf{W}\mathbf{x}, \tag{3.2}$$

and on the other hand

$$\mathbf{A}\mathbf{x} = \lambda \mathbf{x} = \sum_{i=1}^{n} t_i \lambda \mathbf{y}_i. \tag{3.3}$$

Equating the right-hand sides of (3.2) and (3.3) we get that

$$\sum_{i=1}^{k} t_i (\lambda - \beta_i) \mathbf{y}_i + \sum_{i=k+1}^{n} t_i \lambda \mathbf{y}_i = \mathbf{W}\mathbf{x}.$$

Applying the Pythagorean Theorem

$$\sum_{i=1}^{k} t_i^2 (\lambda - \beta_i)^2 + \sum_{i=k+1}^{n} t_i^2 \lambda^2 = \|\mathbf{W}\mathbf{x}\|^2 = \mathbf{x}^{\mathrm{T}} \mathbf{W}^{\mathrm{T}} \mathbf{W} \mathbf{x} \leqslant \varepsilon^2, \tag{3.4}$$

as $\|\mathbf{x}\| = 1$ and the largest eigenvalue of $\mathbf{W}^{\mathrm{T}}\mathbf{W}$ is $\varepsilon^2$.

The squared distance between $\mathbf{x}$ and $F$ is $\text{dist}^2(\mathbf{x}, F) = \sum_{i=k+1}^{n} t_i^2$. As $|\lambda| \geqslant \Delta - \varepsilon$,

$$(\Delta-\varepsilon)^2\mathrm{dist}^2(\mathbf{x},F)=(\Delta-\varepsilon)^2\sum_{i=k+1}^{n}t_i^2\leqslant\sum_{i=k+1}^{n}t_i^2\lambda^2$$
$$\leqslant\sum_{i=1}^{k}t_i^2(\lambda-\beta_i)^2+\sum_{i=k+1}^{n}t_i^2\lambda^2\leqslant\varepsilon^2,$$

where in the last inequality we used (3.4). From here

$$\mathrm{dist}^2(\mathbf{x},F)\leqslant\frac{\varepsilon^2}{(\Delta-\varepsilon)^2}=\mathrm{O}\left(\frac{1}{n}\right) \tag{3.5}$$

follows.

Applying (3.5) for the eigenvectors $\mathbf{x}_1,\dots,\mathbf{x}_k$ of $\mathbf{A}$ and adding the $k$ inequalities together we obtain the same order of magnitude for the sum of the squared distances. □

Now let $G=(V,\mathbf{A})$ be a random weighted graph on an $n$-element vertex set $V$ and with $n\times n$ symmetric weight matrix $\mathbf{A}$ that is a noisy random matrix obtained by adding a Wigner-noise $\mathbf{W}$ to the blown up matrix $\mathbf{B}$. Let us denote by $V_1,\dots,V_k$ the partition of $V$ with respect to the blow-up (it also defines a clustering of the vertices). Proposition 3.1 implies the well-clustering property of the representatives of the vertices of $G$ in the following representation. Let $\mathbf{X}$ be the $n\times k$ matrix containing the eigenvectors $\mathbf{x}_1,\dots,\mathbf{x}_k$ in its columns. Let the $k$-dimensional representatives of the vertices be the row vectors of $\mathbf{X}$: $\mathbf{x}^{(1)},\dots,\mathbf{x}^{(n)}\in\mathbb{R}^k$. Let $S_k^2(\mathbf{X})$ denote the $k$-variance—introduced in [4]—of these representatives in the clustering $V_1,\dots,V_k$:

$$S_k^2(\mathbf{X})=\sum_{i=1}^{k}\sum_{j\in V_i}\|\mathbf{x}^{(j)}-\bar{\mathbf{x}}^{(i)}\|^2,\quad\text{where}\quad\bar{\mathbf{x}}^{(i)}=\frac{1}{n_i}\sum_{j\in V_i}\mathbf{x}^{(j)}. \tag{3.6}$$

**Theorem 3.2.** *With the above notation for the $k$-variance of the representation of the noisy weighted graph $G=(V,\mathbf{A})$ the relation*

$$S_k^2(\mathbf{X})=\mathrm{O}\left(\frac{1}{n}\right)$$

*holds true almost surely.*

**Proof.** By Theorem 3 of [5] it can easily be seen that $S_k^2(\mathbf{X})$ is equal to the left-hand side of (3.1), therefore it is also of order $\mathrm{O}(1/n)$. □

Hence, the addition of any kind of a Wigner-noise to a weight matrix that has a blown up structure $\mathbf{B}$ will not change the order of the protruding eigenvalues of the noisy weight matrix, and the block structure of $\mathbf{B}$ can be concluded from the representatives of the vertices (where the representation is performed by means of the corresponding eigenvectors).

With an appropriate Wigner-noise we can also reach that our matrix $\mathbf{B}+\mathbf{W}$ contains 1's in the $(i,j)$th block with probability $p_{ij}$, and 0's otherwise. That is, for indices $1 \leqslant i < j \leqslant k$ and $l \in V_i$, $m \in V_j$ let

$$w_{lm} := \begin{cases} 1-p_{ij} & \text{with probability } p_{ij} \\ -p_{ij} & \text{with probability } 1-p_{ij} \end{cases}$$

be independent random variables, and for $i = 1, \dots, k$ and $l, m \in V_i$ $(l \leqslant m)$ let

$$w_{lm} := \begin{cases} 1-p_{ii} & \text{with probability } p_{ii} \\ -p_{ii} & \text{with probability } 1-p_{ii} \end{cases}$$

be also independent, otherwise $\mathbf{W}$ is symmetric. This $\mathbf{W}$ satisfies the conditions of Definition 1.1 with entries of zero expectation and bounded variance

$$\sigma^2 = \max_{1\leqslant i\leqslant j\leqslant k} p_{ij}(1-p_{ij}) \leqslant \frac{1}{4}.$$

So, the noisy weighted graph $G = (V, \mathbf{B}+\mathbf{W})$ becomes a usual random graph that has an edge between vertices of $V_i$ and $V_j$ with probability $p_{ij}$, $1 \leqslant i \leqslant j \leqslant k$. In particular, the noisy graph with underlying structure $\mathbf{B}$ of Proposition 2.2 has no edges within $V_i$ $(i = 1, \dots, k)$, and it has an edge between vertices of $V_i$ and $V_j$ with the same probability $p = p_{ij}$ $(i \neq j)$. In this case Theorems 2.3 and 3.2 guarantee the existence of $k$ protruding eigenvalues of the incidence matrix of this random graph, while the corresponding eigenvectors give rise to a Euclidean representation of the vertices revealing the vertex sets $V_1, \dots, V_k$.

## 4. Can the skeleton be recognized?

At the end of the previous section we saw that a seemingly completely random 0–1 matrix can have an easily describable linear structure behind it. The question naturally arises: what kind of random matrices have a blown up matrix as a skeleton with a "small" perturbation? The following theorem proves that under very general conditions a random matrix has at least one eigenvalue greater than of order $\sqrt{n}$.

**Theorem 4.1.** *Let* $\mathbf{A}$ *be an* $n \times n$ *random symmetric matrix such that* $0 \leqslant a_{ij} \leqslant 1$ *and the entries are independent for* $i \leqslant j$. *Further let us suppose that there are positive constants* $c_1$ *and* $c_2$ *and* $0 < \delta \leq \Delta \leqslant 1/2$ *such that with the notation* $X_i = \sum_{j=1}^{n} a_{ij}$

$$\mathbb{E}(X_i) \geqslant c_1 n^{\frac{1}{2}+\delta}, \quad \operatorname{Var}(X_i) \leqslant c_2 n^{\frac{1}{2}+\Delta} \quad (i = 1, \dots, n).$$

*Then for any* $0 < \varepsilon < \delta$:

$$\lim_{n\to\infty} \mathbb{P}\left(\lambda_{\max}(\mathbf{A}) \geqslant c_1 n^{\frac{1}{2}+\varepsilon}\right) = 1,$$

*where the constants* $\delta$ *and* $\Delta$ *are only responsible for the speed of the convergence.*

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Remark that the above conditions automatically hold true if there is a constant $0 < \mu_0 < 1$ such that $\mathbb{E}(a_{ij}) \geqslant \mu_0$ for all $i, j$ pairs. This is the case in the theorems of Juhász [12] and Füredi–Komlós [10]. In our case there can be a lot of zero entries, we require only that in each row there are at least $c_1 n^{1/2+\delta}$ entries with expectation greater than or equal to any small fixed positive constant $\mu_0$. As the matrix is symmetric it also holds for the columns. Therefore among the $n^2$ entries there must be at least $\Theta(n^{1+2\delta})$ ones (but not anyhow) with expectation at least a fixed $0 < \mu_0 < 1$, all the others can be zeroes.

To prove the theorem we will need the following lemma.

**Lemma 4.2** (Chernoff inequality for large deviations). *Let $X_1, \ldots, X_n$ be independent random variables, $|X_i| \leqslant K$, $X := \sum_{i=1}^n X_i$. Then for any $a > 0$:*

$$\mathbb{P}\left(|X - \mathbb{E}(X)| > a\right) \leqslant \mathrm{e}^{-\frac{a^2}{2(\mathrm{Var}(X)+Ka/3)}}.$$

**Proof of Theorem 4.1**. As a consequence of the Perron–Frobenius theorem $\lambda_{\max}(\mathbf{A}) \geqslant \min_i X_i$, hence

$$\mathbb{P}\left(\lambda_{\max}(\mathbf{A}) \geqslant c_1 n^{\frac{1}{2}+\varepsilon}\right) \geqslant \mathbb{P}\left(\min_i X_i \geqslant c_1 n^{\frac{1}{2}+\varepsilon}\right),$$

and it is enough to prove that the latter probability tends to 1 ($n \to \infty$). We shall prove that the probability of the complement event tends to 0:

$$\mathbb{P}\left(\text{for at least one } i : X_i < c_1 n^{\frac{1}{2}+\varepsilon}\right) \leqslant n\mathbb{P}\left(\text{for a general } i : X_i < c_1 n^{\frac{1}{2}+\varepsilon}\right). \tag{4.1}$$

From now on we shall drop the suffix $i$ and $X$ denotes the sum of the entries in an arbitrary row of $\mathbf{A}$. As $X$ is the sum of $n$ independent random variables satisfying the conditions of Lemma 4.2 with $K = 1$,

$$\begin{aligned}
\mathbb{P}\left(X < c_1 n^{\frac{1}{2}+\varepsilon}\right) &= \mathbb{P}\left(\mathbb{E}(X) - X > \mathbb{E}(X) - c_1 n^{\frac{1}{2}+\varepsilon}\right) \\
&\leqslant \mathbb{P}\left(|X - \mathbb{E}(X)| > \mathbb{E}(X) - c_1 n^{\frac{1}{2}+\varepsilon}\right) \\
&\leqslant \mathbb{P}\left(|X - \mathbb{E}(X| > c_1 n^{\frac{1}{2}}(n^{\delta} - n^{\varepsilon})\right) \\
&\leqslant \mathrm{e}^{-\frac{c_1^2 n (n^{\delta}-n^{\varepsilon})^2}{2(c_2 n^{\frac{1}{2}+\Delta} + n^{\frac{1}{2}}(n^{\delta}-n^{\varepsilon})/3)}} \\
&\leqslant \mathrm{e}^{-c_3 n^{\frac{1}{2}} \frac{(n^{\delta}-n^{\varepsilon})^2}{n^{\Delta}}} \\
&= \mathrm{e}^{-c_3 n^{\frac{1}{2}-\Delta}(n^{\delta}-n^{\varepsilon})^2}
\end{aligned}$$

with some positive constant $c_3$, in view of the inequalities $0 < \varepsilon < \delta \leqslant \Delta \leqslant 1/2$. Thus the right-hand side of (4.1) can be estimated from above by

$$\frac{n}{\mathrm{e}^{c_3 n^{\frac{1}{2}-\Delta}(n^{\delta}-n^{\varepsilon})^2}} \leqslant \frac{n}{\mathrm{e}^{c_4 n^{\gamma}}}$$

with some $c_4 > 0$ and $\gamma > 0$ because of the previous inequalities for $\varepsilon, \delta, \Delta$. The last term above tends to 0 ($n \to \infty$) that finishes the proof. □

**Theorem 4.3.** *If the $n \times n$ random weight matrix $\mathbf{A}$—with properties in Theorem 4.1—of the random graph $G = (V, \mathbf{A})$ has exactly $k$ eigenvalues of order greater than $\sqrt{n}$, and there is a $k$-partition of the vertices such that the $k$-variance of the representatives is $\mathrm{O}(1/n)$—in the representation with the corresponding eigenvectors—then almost surely there is a blown up matrix $\mathbf{B}$ such that $\mathbf{A} = \mathbf{B} + \mathbf{E}$ with $\|\mathbf{E}\| = \mathrm{O}(\sqrt{n})$.*

**Proof.** Let $\mathbf{x}_1, \dots, \mathbf{x}_k$ denote the eigenvectors corresponding to $\lambda_1, \dots, \lambda_k$, the $k$ largest (of order larger than $\sqrt{n}$) eigenvalues of $\mathbf{A}$. The representatives—that are row vectors of the $n \times k$ matrix $\mathbf{X} = (\mathbf{x}_1, \dots, \mathbf{x}_k)$— by the supposition of the theorem form $k$ clusters in $\mathbb{R}^k$ with $k$-variance less than $c/n$ with some constant $c$. Let $V_1, \dots, V_k$ denote the clusters (properly reordering the rows of $\mathbf{X}$, together they give the index set $\{1, \dots, n\}$). Let $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)} \in \mathbb{R}^k$ be the Euclidean representatives of the vertices (the rows of $\mathbf{X}$), and let $\bar{\mathbf{x}}^{(1)}, \dots, \bar{\mathbf{x}}^{(k)}$ denote the cluster centers, see (3.6). Now let us choose the following representation of the vertices. The representatives are row vectors of the $n \times k$ matrix $\widetilde{\mathbf{X}}$ such that the first $n_1$ rows of $\widetilde{\mathbf{X}}$ be equal to $\bar{\mathbf{x}}^{(1)}, \dots,$ and the last $n_k$ rows of $\widetilde{\mathbf{X}}$ be equal to $\bar{\mathbf{x}}^{(k)}$. Finally, let $\mathbf{y}_1, \dots, \mathbf{y}_k \in \mathbb{R}^n$ be the column vectors of $\widetilde{\mathbf{X}}$. By the considerations of Theorem 3.2

$$S_k^2(\mathbf{X}) = \sum_{i=1}^{k} \mathrm{dist}^2(\mathbf{x}_i, F) < c/n,$$

where the $k$-dimensional subspace $F$ is spanned by the vectors $\mathbf{y}_1, \dots, \mathbf{y}_k$.

Then a set $\mathbf{v}_1, \dots, \mathbf{v}_k$ of orthonormal vectors within $F$ can be found such that

$$\sum_{i=1}^{k} \|\mathbf{x}_i - \mathbf{v}_i\|^2 \leqslant 2\frac{c}{n}$$

holds almost surely, see Proposition 2 of [5]. (We shall use that $\mathbf{v}_i$'s also have equal coordinates within the blocks.) For them

$$\mathbf{x}_i = \sum_{j=1}^{k} t_{ij}\mathbf{v}_j + \mathbf{r}_i,$$

$$\|\mathbf{x}_i - \mathbf{v}_i\|^2 = \|\mathbf{x}_i\|^2 + \|\mathbf{v}_i\|^2 - 2\mathbf{x}_i^{\mathrm{T}}\mathbf{v}_i = 2(1 - t_{ii}) = \mathrm{O}(1/n),$$

therefore

$$\|\mathbf{x}_i - t_{ii}\mathbf{v}_i\|^2 = 1 - t_{ii}^2 = \mathrm{O}(1/n),$$

that implies $|t_{ij}| = \mathrm{O}(1/\sqrt{n})$, $j \neq i$ and $\|\mathbf{r}_i\|^2 = \mathrm{O}(1/n)$.

Hence

$$\begin{aligned}\sum_{i=1}^{k}\lambda_i\mathbf{x}_i\mathbf{x}_i^{\mathrm{T}} &= \sum_{i=1}^{k}\lambda_i\left(\sum_{j=1}^{k}t_{ij}\mathbf{v}_j+\mathbf{r}_i\right)\left(\sum_{j=1}^{k}t_{ij}\mathbf{v}_j^{\mathrm{T}}+\mathbf{r}_i^{\mathrm{T}}\right)\\ &= \sum_{i=1}^{k}\lambda_i\mathbf{v}_i\mathbf{v}_i^{\mathrm{T}}-\sum_{i=1}^{k}\lambda_i(1-t_{ii}^2)\mathbf{v}_i\mathbf{v}_i^{\mathrm{T}}+\sum_{i=1}^{k}\lambda_i\sum_{j\neq i}t_{ij}^2\mathbf{v}_j\mathbf{v}_j^{\mathrm{T}}\\ &\quad+\sum_{i=1}^{k}\lambda_i\left(\sum_{j\neq i}\left(t_{ii}t_{ij}\mathbf{v}_i\mathbf{v}_j^{\mathrm{T}}+t_{ij}t_{jj}\mathbf{v}_j\mathbf{v}_i^{\mathrm{T}}\right)+\sum_{j\neq i}\sum_{l\neq i}t_{ij}t_{il}\mathbf{v}_j\mathbf{v}_l^{\mathrm{T}}\right)\\ &\quad+\sum_{i=1}^{k}\lambda_i\left(\sum_{j=1}^{k}t_{ij}\mathbf{r}_i\mathbf{v}_j^{\mathrm{T}}+\sum_{j=1}^{k}t_{ji}\mathbf{v}_j\mathbf{r}_i^{\mathrm{T}}+\mathbf{r}_i\mathbf{r}_i^{\mathrm{T}}\right).\end{aligned} \tag{4.2}$$

With the triangle inequality the norm of the left-hand side matrix can be estimated from above with the sum of the norms of the individual terms. First we estimate the squared norms and use that $\lambda_i^2=\mathrm{O}(n^{1+2\varepsilon})$, $1-t_{ii}^2=\mathrm{O}(1/n)$ and $\|\mathbf{r}_i\|^2=\mathrm{O}(1/n)$, further

$$\|\mathbf{v}_i\mathbf{v}_j^{\mathrm{T}}\|^2=\|\mathbf{v}_i\mathbf{v}_j^{\mathrm{T}}(\mathbf{v}_i\mathbf{v}_j^{\mathrm{T}})^{\mathrm{T}}\|=\|\mathbf{v}_i\mathbf{v}_i^{\mathrm{T}}\|=\mathbf{v}_i^{\mathrm{T}}\mathbf{v}_i=1$$

and similarly,

$$\|\mathbf{r}_i\mathbf{v}_j^{\mathrm{T}}\|^2=\|\mathbf{r}_i\mathbf{v}_j^{\mathrm{T}}(\mathbf{r}_i\mathbf{v}_j^{\mathrm{T}})^{\mathrm{T}}\|=\|\mathbf{r}_i\mathbf{r}_i^{\mathrm{T}}\|=\mathbf{r}_i^{\mathrm{T}}\mathbf{r}_i\leqslant\frac{a}{n}$$

with some constant $a$. For details, see the proof of Theorem 4 in [5].

Summarizing, as $\mathbf{A}=\sum_{i=1}^{n}\lambda_i\mathbf{x}_i\mathbf{x}_i^{\mathrm{T}}$ and the spectral norm of the part $\sum_{i=k+1}^{n}\lambda_i\mathbf{x}_i\mathbf{x}_i^{\mathrm{T}}$ is at most $\sqrt{n}$, we can choose $\mathbf{B}=\sum_{i=1}^{k}\lambda_i\mathbf{v}_i\mathbf{v}_i^{\mathrm{T}}$—the first term in (4.2)—for the blown up matrix, while the norm of the remaining terms—they, together with $\sum_{i=k+1}^{n}\lambda_i\mathbf{x}_i\mathbf{x}_i^{\mathrm{T}}$, will form $\mathbf{E}$—is estimated from above by $n^{\varepsilon}$ with $\varepsilon<1/2$, that finishes the proof. □

## 5. Conclusions and other directions

In the models discussed in Sections 2 and 3 a special kind of a random noise was added to a fairly general underlying structure. We have shown that if the adjacency matrix of our underlying graph on $n$ vertices has some protruding eigenvalues (of order $n$ in absolute value), then a Wigner-noise cannot disturb essentially this structure: the adjacency matrix of the noisy graph will have the same number of protruding eigenvalues with corresponding eigenvectors revealing the structure of the graph. Vice versa, if the representation with them shows well metric classification properties, in Section 4 we have shown, how to find the clusters themselves.

Theoretically, for any graph on $n$ vertices, the Regularity Lemma of Szemerédi guarantees the existence of a partition $V_0, V_1, \dots, V_k$ of the vertices (here $V_0$ is a "small" exceptional set) such that the edge-densities between most of the $V_i, V_j$ pairs ($1 \leqslant i < j \leqslant k$) are homogeneous in the following sense. We say that a pair $V_i, V_j$ ($i \neq j$) is $\varepsilon$-regular, if for any $A \subset V_i$, $B \subset V_j$ with $|A| > \varepsilon|V_i|$, $|B| > \varepsilon|V_j|$ $|\mathrm{dens}\,(A, B) - \mathrm{dens}\,(V_i, V_j)| < \varepsilon$ holds, where dens $(A, B)$ denotes the edge-density between the disjoint vertex sets $A$ and $B$. In fact, denoting by cut $(A, B)$ the cut-set between $A$ and $B$,

$$\mathrm{dens}\,(A, B) = \frac{|\mathrm{cut}\,(A, B)|}{|A| \cdot |B|}.$$

If the graph is sparse—the number of edges $e = \mathrm{O}(n^2)$—then $k = 1$, otherwise $k$ can be arbitrarily large (but it depends only on $\varepsilon$).

If our random graph has a blown up skeleton, then $|\mathrm{cut}\,(V_i, V_j)|$ is the sum of $|V_i| \cdot |V_j|$ independent, identically distributed Bernoulli variables with parameter $p_{ij}$ ($1 \leqslant i, j \leqslant k$), where $p_{ij}$'s are entries of the pattern matrix $\mathbf{P}$. Hence $|\mathrm{cut}\,(A, B)|$ is a binomially distributed random variable with expectation $|A| \cdot |B| \cdot p_{ij}$ and variance $|A| \cdot |B| \cdot p_{ij}(1 - p_{ij})$. Therefore by Lemma 4.2 (with the choice $K = 1$) and with $A \subset V_i$, $B \subset V_j$, $|A| > \varepsilon|V_i|$, $|B| > \varepsilon|V_j|$ we have that

$$\begin{aligned}\mathbb{P}\left(|\mathrm{dens}\,(A, B) - p_{ij}| > \varepsilon\right) &= \mathbb{P}\left(\left||\mathrm{cut}\,(A, B)| - |A| \cdot |B| \cdot p_{ij}\right| > \varepsilon \cdot |A| \cdot |B|\right)\\ &\leqslant \mathrm{e}^{-\frac{\varepsilon^2 |A|^2 |B|^2}{2[|A||B| p_{ij}(1-p_{ij}) + \varepsilon|A||B|/3]}}\\ &= \mathrm{e}^{-\frac{\varepsilon^2 |A||B|}{2[p_{ij}(1-p_{ij}) + \varepsilon/3]}}\\ &\leqslant \mathrm{e}^{-\frac{\varepsilon^4 |V_i||V_j|}{2[p_{ij}(1-p_{ij}) + \varepsilon/3]}},\end{aligned}$$

that tends to 0, as $|V_i| = n_i \to \infty$ and $|V_j| = n_j \to \infty$. Hence, any pair $V_i, V_j$ is almost surely $\varepsilon$-regular. In this case our random graph turns out to be a so-called generalized random graph of [13], that is the sum of a blown-up skeleton and a noise. We note, however, that the Regularity Lemma does not give a construction for the clusters. Provided the conditions of Theorem 4.3 hold, by the cluster centers a similar construction is given in the proof of the theorem. Some algorithmic aspects of the Regularity Lemma are also discussed in [9].

In fact, there are other kind of real-world graphs that are more or less vulnerable to random noise, e.g. scale-free graphs introduced in [3]. Bollobás and Riordan [7] investigate the vulnerability of this graph under the effect of removing edges, if $n \to \infty$. In the sequel I shall use the definition of Chung et al. [8] for a graph on $n$ vertices with given positive expected degree sequence $d_1, \dots, d_n$. Let $d_{ij} := d_i d_j / \sum_{l=1}^{n} d_l$ be the weight of the connection between the $i$th and $j$th vertices, where loops are also present and we suppose that $\max_i d_i^2 \leqslant \sum_{i=1}^{n} d_i$. So our weight matrix $\mathbf{D} =$

$(d_{ij})_{i,j=1}^{n}$ is a diadic product, having the eigenvalue zero with multiplicity $n-1$, further the only positive eigenvalue is equal to

$$\frac{\sum_{i=1}^{n} d_i^2}{\sum_{i=1}^{n} d_i}, \tag{5.1}$$

the second order average degree introduced in [8]. In my approach the random noise means the addition of a Wigner-noise to **D**, the effect of which depends on the asymptotic order of the quantity (5.1).

The random power law graph is a special case of this model. Let $\beta > 0$ denote the power in the distribution of the actual degrees: the probability that a vertex has degree $x$ is proportional to $1/x^{\beta}$ ($x$ is not necessarily an integer). The maximum eigenvalue of our graph is proportional to the square root of the maximum degree, see [8]. Móri [14] proves that in case of trees the maximum degree is asymptotically of order $n^{1/(\beta-1)}$, if $n$ is "large", and this asymptotic order is also valid for other power law graphs with $\beta > 1$. Hence, with $1/2(\beta-1) > 1/2$, that is with $\beta < 2$ the largest eigenvalue has order greater than $\sqrt{n}$ that is not changed significantly after a Wigner-noise is added.

In view of [8] the following degree sequence gives a power law graph with parameters $\beta$ (the power) and $i_0$ (specifies the support of the distribution):

$$d_i = c \cdot i^{-\frac{1}{\beta-1}}, \quad i = i_0, \ldots, i_0 + n,$$

where $c$ is a normalizing constant.

In order to have a real graph the following two inequalities must hold:

$$\sum_{i=i_0}^{i_0+n} d_i = 2e \leqslant 2\binom{n+1}{2} = (n+1)n \sim n^2, \tag{5.2}$$

where $e$ denotes the number of edges, and for the minimum degree

$$d_{\min} = d_{i_0+n} = c \cdot (i_0+n)^{-\frac{1}{\beta-1}} \geqslant 1. \tag{5.3}$$

For large $n$ the sum $\sum_{i=i_0}^{i_0+n} d_i$ is bounded by means of integration, hence the left-hand side of (5.2) is estimated as

$$\begin{aligned}\sum_{i=i_0}^{i_0+n} d_i &= c\sum_{i=i_0}^{i_0+n} i^{-\frac{1}{\beta-1}} \geqslant c\int_{i=i_0}^{i_0+n-1} x^{-\frac{1}{\beta-1}}\,\mathrm{d}x \\ &= c\frac{\beta-1}{2-\beta}\left[i_0^{-\frac{\beta-1}{2-\beta}} - (i_0+n-1)^{-\frac{\beta-1}{2-\beta}}\right],\end{aligned} \tag{5.4}$$

where $1 < \beta < 2$.

Relations (5.2)–(5.4) give upper and lower estimates for $c$:

$$(i_0+n)^{\frac{1}{\beta-1}} \leqslant c \leqslant \frac{n^2}{\frac{\beta-1}{2-\beta}\left[i_0^{-\frac{\beta-1}{2-\beta}} - (i_0+n-1)^{-\frac{\beta-1}{2-\beta}}\right]} = \mathrm{O}(n^2)$$

for large $n$'s. This surely holds, if $1/(\beta - 1) \leqslant 2$, that is, if $\beta \geqslant 1.5$. If, in addition, $\beta < 2$ holds, the largest eigenvalue is greater than $\sqrt{n}$ in magnitude. Consequently, for $\beta \in [1.5, 2)$ $c$ can be chosen such that the number of edges $e = \Theta(n^2)$, so our graph is dense enough to have more than one cluster by the Regularity Lemma. In other words, our graph has a blown up skeleton and, therefore, it is robust enough. For example, $\beta$ is 1.5 in the flux distribution examined in [2]. Scale-free graphs with $\beta \in [1.5, 2)$ frequently occur in case of cellular networks. Perhaps, because of this, such metabolic networks can better tolerate a Wigner-noise—that more or less affects each of the edges—than those with $\beta \geqslant 2$, usual in case of social and communication networks.

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